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# Analytical model for a geomorphological instantaneous unit hydrograph

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## Abstract:

The rainfall-runoff modelling of a river basin can be divided into two processes: the production function and the transfer function. The production function determines the proportion of gross rainfall actually involved in the runoff. The transfer function spreads the net rainfall over time and space in the river basin. Such a transfer function can be modelled using the approach of the geomorphological instantaneous unit hydrograph (GIUH). The effectiveness of geomorphological models is actually revealed in rainfall-runoff modelling, where hydrologic data are desperately lacking, just as in ungauged basins. These models make it possible to forecast the hydrograph shape and runoff variation versus time at the basin outlet. This article is an introduction to a new GIUH model that proves to be simple and analytical. Its geomorphological parameters are easily available on a map or from a digital elevation model. This model is based on general hypotheses on symmetry that provide it with multiscale versatile characteristics. After having validated the model in river basins of very different nature and size, we present an application of this model for rainfall-runoff modelling. Since parameters are determined relying on real geomorphological data, no calibration is necessary, and it is then possible to carry out rainfall-runoff simulations in ungauged river basins. Copyright © 2006 John Wiley & Sons, Ltd.

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## INTRODUCTION

The purpose of rainfall-runoff modelling is the conversion of rainfall into runoff at the river basin outlet. This complex process is the result of the combination of multiple factors involved at various spatial scales. The process pattern made by hydrologists usually includes two functions: the production function and the transfer function. The production function makes it possible to determine the fallen rain (gross rainfall) actually involved in the runoff at the river outlet. The transfer function works out the runoff at the outlet according to the net rainfall. Here, we will focus exclusively on the transfer function and its geomorphological characteristics.

The transfer function can be understood through an overall approach (Chow *et al.*, 1988), namely instantaneous unit hydrograph (IUH) theory (Sherman, 1932; Nash, 1957, Dooge, 1959; Lienhard, 1964; Rodriguez-Iturbe and Valdès, 1979; Duchesne *et al.*, 1997). IUH theory (Sherman, 1932) is a determinist model that relies on three major hypotheses:

- a univocal relationship between rainfall and runoff, in the case of rainfall having the same intensity and duration features;
- linearity between unit rainfalls of different intensity and the corresponding runoffs;
- uniformity of rainfall over the whole surface of the river basin.

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The three hypotheses are quite restrictive. However, the unit hydrograph method has a major advantage, i.e. its simplicity; this is why it has been used for many years on quite a large number of river basins throughout the world (Chow *et al.*, 1988). The unit hydrograph method is based on three basic principles:

- The unit rainfall is a rainfall impulse by definition, which means that it is short lasting compared with the basin lag, homogeneous in space and with a sufficient intensity to generate a runoff surge.
- For a unit rainfall involving a runoff depth  $h$  (m), the relation between the rate of flow by unit area  $q(t)$  ( $\text{m s}^{-1}$ ), the unit hydrograph  $T(t)$  ( $\text{s}^{-1}$ ) and  $h$  is:

$$q(t) = h(t)T(t) \quad (1)$$

- The additionality property applying to the unit hydrograph is a linear element. Therefore, if a complex rainfall is divided into a succession of unit rainfalls, it is possible to calculate the corresponding hydrograph as follows:

$$q(t) = \int_0^t i(\tau)T(t - \tau)d\tau \quad (2)$$

where  $q(t)$  ( $\text{m s}^{-1}$ ) is the rate of flow per unit area,  $i(t)$  ( $\text{m s}^{-1}$ ) is the net rainfall (rainfall in excess of infiltration and interception) falling uniformly on the drainage basin, and  $T(t)$  ( $\text{s}^{-1}$ ) is a transfer function, and more precisely the probability density function of time transfer on the drainage basin (Gupta *et al.*, 1980). Consequently, the difficulties of expressing the rate of flow  $q(t)$  are twofold: what is the value of the net rainfall  $i(t)$  and which function  $T(t)$  should be chosen?

Choosing the characteristic IUH of a river basin is a delicate issue. Its experimental measurement with real rainfall events is always worked out in gauged river basins, but it remains difficult to carry out. Consequently, theoretical approaches for identifying a suitable IUH have been used. For example, (Nash, 1957, 1959, 1960) conceptualized the river basin and compared it with a series of  $n$  identical linear reservoirs, whose analytical model for the unit hydrograph is a two-parameter Gamma law, where  $n$  is the number of cascade reservoirs and  $\tau$  is the usual release rate from one reservoir into another.

The  $n$  and  $\tau$  parameters have no physical reality and must be determined by relying on previous accounts of rainfall-runoff events. This is one of the major drawbacks of the Nash model and other similar models (Dooge, 1959; Lienhard, 1964), which cannot, therefore, be used in ungauged river basins.

If the hydrograph depends on rainfall, it also depends strongly on the geomorphological features of the river basin (shape, slope, organization of the river network). Therefore, determining a geomorphological transfer function such as the geomorphological IUH (GIUH; Rodriguez-Iturbe and Valdès, 1979), or the model described in the present article, makes it possible to give physical meaning to the unit hydrograph that has been constructed on the basis of parameters set according to the river basin's geometry. Such geomorphological models are based on the analyses of the morphological characteristics of river basins and, more particularly, river networks.

The purpose of this article is to present a new analytical model for the geomorphological transfer function, i.e. a theoretical law giving the distribution of hydraulic pathways according to the river basin geomorphology. This model has already been introduced in Fleurant and Boulestreau (2005) and is a theoretical coupling of quantitative geomorphology and hydrology. Here, we present some applications of the model. We will compare this theoretical law with the distribution of real hydraulic pathways in two river basins. By way of an applied example, we will also show how to use this geomorphological transfer function as an analytical GIUH (AGIUH), and thus make rainfall-runoff simulations.

## AN ANALYTICAL GEOMORPHOLOGICAL TRANSFER FUNCTION

The geomorphological theory of the unit hydrograph has been formulated by Rodriguez-Iturbe and Valdès (1979). This model is called GIUH, and provides the times of transfer according to the geomorphology of the

drainage basin. It can be expressed as follows (Rodriguez-Iturbe and Rinaldo, 1997):

$$T(t) = \sum_{\gamma \in \Gamma} p(\gamma) f_{\gamma}(t) \quad (3)$$

where  $p(\gamma)$  is the probability to have a  $\gamma$  path,  $\gamma$  being any path covered by a raindrop in the stream network, and  $\Gamma$  is the set of all the possible paths  $\gamma$ .  $f_{\gamma}(t)$  ( $s^{-1}$ ) is the probability density function of the travel time of the path  $\gamma$ .

The GIUH model  $T(t)$  in Equation (3) is the frequency distribution of travel times from points within the drainage basin to the outlet. Rosso (1984) shows that Equation (3) can be fitted by a Gamma law whose parameters depend on Horton's parameters.

The GIUH model has generated a wealth of literature over the last 25 years. For most modifications, workers put forward the hypothesis that the response to rainfall input coming out of each drain is either an exponential function (Rodriguez-Iturbe and Valdès, 1979), a single-valued function (Gupta *et al.*, 1980; Gupta and Waymire, 1983), or else a Gamma function (van der Tak and Bras, 1990; Jin, 1992; Kirchner *et al.*, 2001). In all cases, these hypotheses are empirical and, therefore, influenced by experimental observations.

The model described in this article is different from the GIUH of Rodriguez-Iturbe and Valdès (1979) from two main aspects:

- It is based on general hypotheses on symmetry combined with the fractal properties of branched structures; thus, it is able to be used for branched structures in general (Duchesne *et al.*, 2002; Fleurant *et al.*, 2004), and not for stream networks exclusively.
- The reasoning is carried out on pathway lengths rather than on the times of transfer to the basin.

The model introduced here uses the findings of research carried out on stream networks (Duchesne *et al.*, 1997; Cudennec, 2000; Cudennec *et al.*, 2004). The focus is on describing network structures thanks to Strahler stream ordering (Strahler, 1964), and by applying Horton's laws (Horton, 1945) to a geomorphological statistics rationale, in order to elaborate a descriptive theoretical model of network lengths.

Well before Mandelbrot (1988) developed his fractal theory; branching networks, and more particularly stream network structures, had been studied very accurately. In fact, geological scientists and hydrologists such as Horton (1945), Schumm (1956), Strahler (1957) or Shreve (1969) took an interest in analysing the complex ordering of these networks. They made topological and hydraulic analyses (Kirshen and Bras, 1983) that can be applied to all branching networks that are three-dimensional and organized into a hierarchy. Many classification systems have been put forward (Horton, 1945), but we have decided upon Strahler's system (Strahler, 1952), which is the most widely used. The classification system (Figure 1) is as follows:

- headwaters are first-order stream segments;
- when two stream segments within the same order  $i$  merge, the stream segment resulting from this confluence is within order  $i + 1$ ;
- when two stream segments within different orders  $i$  and  $j$  merge, the stream segment resulting from this confluence is within order  $\max(i, j)$ .

Strahler's classification thus makes it possible to organize the different segments of a stream network into a hierarchy. Consequently, the stream outlet will have the highest index value, corresponding to the river network order.

Hydrologists relied on this classification to put forward general geometric laws concerning the ordering of stream networks. Among them, Horton's laws (Horton, 1945) describe the way stream networks are organized. One of these laws expresses the so-called length ratio  $R_L = \bar{l}_i / \bar{l}_{i-1}$ , also known as Horton's ratio. A great number of experimental studies on stream networks (LaBarbera and Rosso, 1987, 1989, 1990; Feder, 1988;

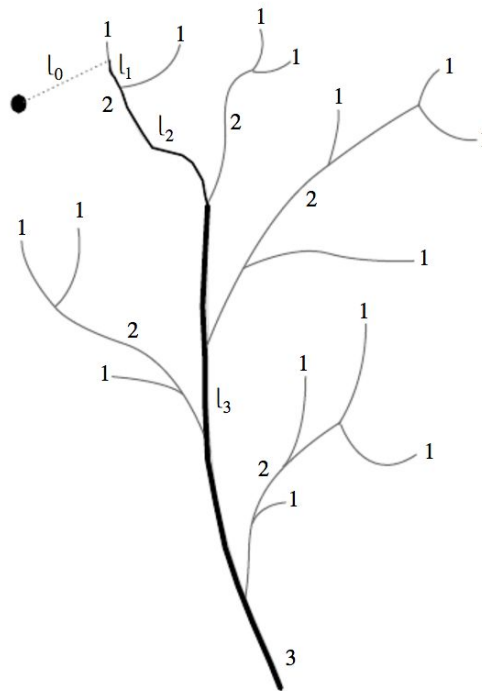


Figure 1. The concept of Strahler's hierarchical classification system. The hydraulic length is the added lengths of channel up to an indefinite raindrop of the river basin. Here, the path between a raindrop of the river basin and the outlet passes through first, second and third orders

Tarboton *et al.*, 1990; Rosso *et al.*, 1991) reveal that this ratio is rather stable and fluctuates between 1.5 and 3.5.

A brief outline of the model is given hereafter, and a detailed description can be found in Appendix A.

#### Definition of hydraulic lengths

Before giving the theoretical expression of probability density function of hydraulic lengths, it is necessary to define this variable accurately. As shown in Figure 1, if we take an indefinite point on the river basin to represent a raindrop, then the path to be covered between this point and the outlet successively goes over channels of increasing orders. The hydraulic length is defined as:

$$L = l_0 + \sum_{k=1}^n l_k \quad (4)$$

where  $l_k$  (m) is the length of the channel in  $k$  order and  $n$  is the order of the river network.  $l_0$  (m) is the length of the hillslope. Hence, the hydraulic length is the added lengths  $n$  of the channels.

The hydraulic length  $L$  (m), in the river network, may be calculated using a vector with  $n$  components  $(l_1, l_2, \dots, l_n)$ . It is important to note that some length can be nil: a first-order channel might join a third-order one directly, so  $l_2$  would be zero in that particular instance.

#### The transfer function of the drains

The rationale of Duchesne *et al.* (1997); Cudennec (2000) and Cudennec *et al.*, (2004), founded on the hypotheses on symmetry combined with fractal geometry, leads to the probability density function of the

drains' hydraulic lengths of order  $k$  that we have reformulated as follows (see Appendix A for details):

$$\text{pdf}(l_k) = \frac{1}{\sqrt{2\pi\bar{l}_k}} \frac{1}{\sqrt{l_k}} e^{-l_k/2\bar{l}_k} \quad k = 1, 2, \dots, n \quad (5)$$

A Gamma distribution with  $\alpha = 1/2$  and  $\beta = 1/2\bar{l}_k$  can be recognized here and termed  $\Gamma(l_k, 1/2, 1/2\bar{l}_k)$ , where  $\bar{l}_k$  is the mean of  $l_k$ .

#### The geomorphological transfer function in the stream network

We put forward the hypothesis that the distribution law in Equation (5) is an appropriate model for representing the hydraulic pathways of individual drains, which is consistent with research work by van der Tak and Bras (1990), Jin (1992) and Kirchner *et al.* (2001).

Now, the step consists of determining the probability density function for  $L$ , knowing that  $L = \sum_{k=1}^n l_k$ . The usual way to calculate the probability density function of  $L$  is to use the following property (Feller, 1971): if we have  $n$  independent random variables  $l_k$ ,  $k = 1, 2, \dots, n$ , of probability density function  $\text{pdf}(l_k)$ , then the random variable  $L$  defined as  $L = \sum_{k=1}^n l_k$  has a probability density function  $\text{pdf}(L)$  such as:

$$\text{pdf}(L) = \text{pdf}(l_1) * \text{pdf}(l_2) * \dots * \text{pdf}(l_n) \quad (6)$$

where  $*$  is the convolution integral. A full explanation of the convolution operator  $*$  is given in Appendix A. Equation (6) can be calculated very easily with numerical calculation software, though an analytical solution exists that is explained in this paper.

The result of the convolution integral for  $n$  Gamma distributions is obtained by working out the generalized characteristic function (Mathai, 1982; Moschopoulos, 1985), and we can thus assert the following theorem (Sim, 1992):

Let  $l_1, l_2, \dots, l_n$  be  $n$  independent random distributed according to a Gamma distribution of the following type  $\Gamma(l_k, 1/2, 1/2\bar{l}_k)$ , then the probability density for  $L = \sum_{k=1}^n l_k$  is:

$$\begin{aligned} \text{pdf}(L) &= \text{pdf}(l_1) * \text{pdf}(l_2) * \dots * \text{pdf}(l_n) \\ &= \Gamma\left(l_1, \frac{1}{2}, \frac{1}{2\bar{l}_1}\right) * \Gamma\left(l_2, \frac{1}{2}, \frac{1}{2\bar{l}_2}\right) * \dots * \Gamma\left(l_n, \frac{1}{2}, \frac{1}{2\bar{l}_n}\right) \\ &= \frac{1}{\sqrt{2^n \prod_{i=1}^n \bar{l}_i}} \frac{L^{(n/2)-1}}{\Gamma\left(\frac{n}{2}\right)} e^{-L/2\bar{l}_n} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left[\frac{L}{2(\bar{l}_n - \bar{l}_{n-1})}\right]^k \end{aligned} \quad (7)$$

with

$$b_i(k) = \begin{cases} 1 & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\frac{i-2}{2}\right)_j (-k)_j}{j! \left(\frac{i-1}{2}\right)_j} \left(\frac{\bar{l}_{i-1} - \bar{l}_{i-2} \bar{l}_{i-1}}{\bar{l}_{i-1} - \bar{l}_i \bar{l}_{i-2}}\right)^j & \text{for } i = 3, 4, \dots, n \end{cases}$$

and

$$(x)_k = x(x+1)(x+2) \dots (x+k-1) \quad (8)$$

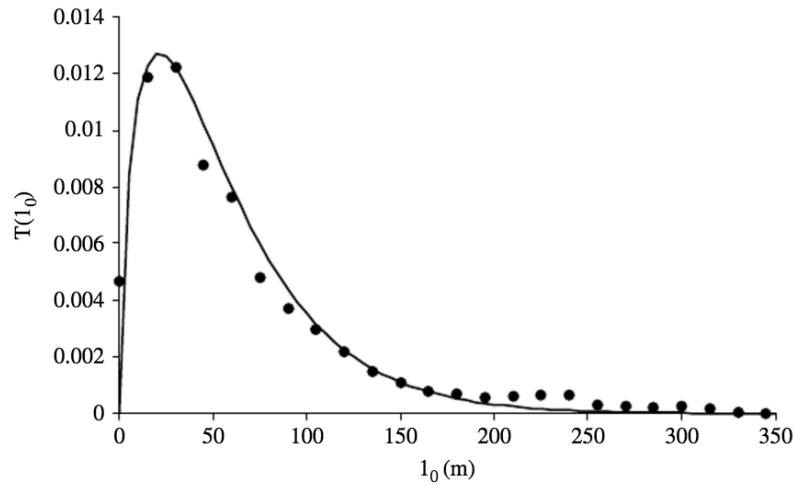


Figure 2. Distributions of the hydraulic lengths on the hillslopes of Bunder and the corresponding probability density function  $\text{pdf}(l_0)$

Equation (7) is a geomorphological transfer analytical function, since it represents the distribution of hydraulic pathways through which water travels inside the river network. Moreover, the parameters of this analytical function are geomorphological:

- the Strahler order  $n$  represents the rate of the hierarchical system of the river network;
- the lengths  $\bar{l}_k$  (m) are the average hydraulic lengths for each Strahler order.

In fact, this transfer function is equivalent to a Width function (Rodriguez-Iturbe and Rinaldo, 1997), since it gives the probability of having a given distance between a definite point of the stream network and the outlet of the drainage basin.

#### *The transfer function applied to the hillslopes*

First, the impact of time transfer applied to the slopes was not considered by the GIUH model. Then, van der Tak and Bras (1990) took this impact into account by convoluting a transfer function on the basin slopes thanks to the GIUH model. The transfer function applied to the basin slopes is a Gamma  $\text{pdf}(l_0) = \Gamma(l_0, \alpha_h, \beta_h)$  law.

The  $\alpha_h$  shape parameter can be determined experimentally and can reach values ranging from 1.2 to 3.1 (van der Tak and Bras, 1990). The value of the scale parameter is  $\beta_h = \alpha/\bar{l}_0$ , so that the  $\Gamma(l_0, \alpha_h, \beta_h)$  integral is still equal to unity (Figure 2).

## METHODOLOGY AND EXPERIMENTAL DATA

In order to compare the geomorphological transfer analytical function in Equation (7) with data from real river basins, it is first necessary to assess the parameter values of the model (Strahler order  $n$  and average lengths  $l_k$  within the different orders  $k$ ). Then we will just have to compare the graph of the function in Equation (7) with that of the experimental Width function of the river basins studied.

#### *Assessment of the model's parameters*

The parameters of the model are of a geomorphological nature and easily available; these are the Strahler order  $n$  of the river basin and the drains' average lengths of order  $k$  ( $\bar{l}_k$ ). These parameters can be assessed either with numerical data from the river basin (digital elevation model, digitalization), combined with retrieval

software for geomorphological data (ArcView, MapWindow), or manually with a map of the stream network. In the second case, the procedure is as follows: first, the river basin is divided into a number of points representing raindrops falling uniformly on the river basin surface. The path of the raindrops down the hillslope to the stream network is determined using the contour lines; it is represented by length  $l_0$  in Figure 1. Finally, the distance from this point to the river basin outlet is assessed by combining the covered lengths with the different Strahler orders; they correspond to lengths  $l_1, l_2, \dots, l_n$ .

For each point (or raindrop) we have the lengths  $l_1, l_2, \dots, l_n$  (some of them are obviously equal to zero) corresponding to the distance covered by a raindrop through the river network, within the different Strahler orders. Thus, we can calculate the average distance  $\bar{l}_1, \bar{l}_2, \dots, \bar{l}_n$  covered for all the raindrops.

A sensitivity study of this methodology was carried out by Kartiwa (2004), revealing that it is not necessary to have a large number of points, i.e. too detailed a grid. In fact, Kartiwa's study on several river basins highlights the fact that shifting from a 3000-point grid to a 30-point grid involves very few measurement errors exceeding 10% of the estimated  $\bar{l}_k$  averages. As for the Strahler order  $n$  of the river basin, it is obtained even more quickly by reading the map and applying the Strahler rules specified before. This sensitivity study highlights the fact that the parameters of the model are manually available quite rapidly.

#### Experimental data

In order to compare the analytical equation of the geomorphological transfer function (Equation (7)) with reality, we have chosen two very different river basins with regard to scale and morphology (Figures 3 and 4).

The Bunder river basin is located in Indonesia, its surface area is  $29 \times 10^4 \text{ m}^2$  and its Strahler order is 3. It is situated in the Wonosari region, 80 km north of Yogyakarta, the province capital,  $07^\circ 53'S$  and  $110^\circ 32'E$ .

The river basin of Saint-Michel Mont-Mercure is located in France (Vendée), its surface area is  $673 \times 10^4 \text{ m}^2$ , and its Strahler order is 5. Its precise location is Vendée,  $46^\circ 48'3''N$  and  $0^\circ 54'8''W$ .

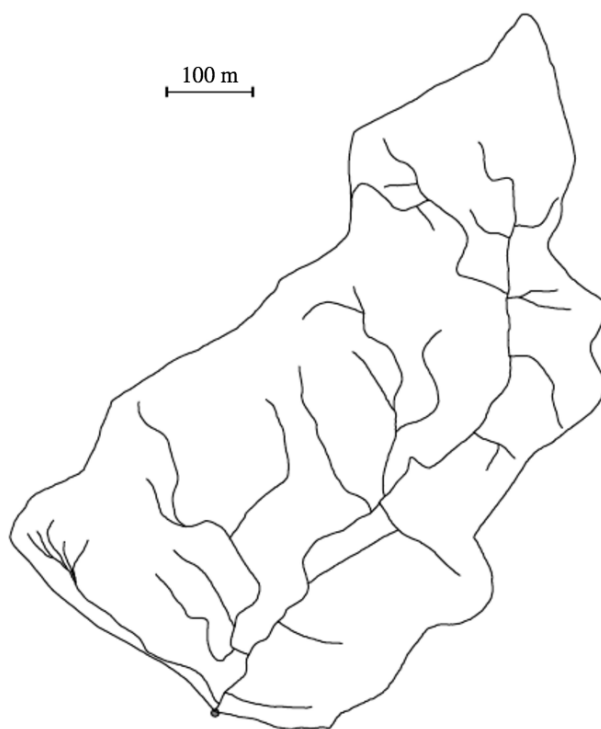


Figure 3. Bunder watershed map, Wonosari, Yogyakarta, Indonesia. The Bunder is a Strahler order 3 river basin with area of only  $29 \times 10^4 \text{ m}^2$

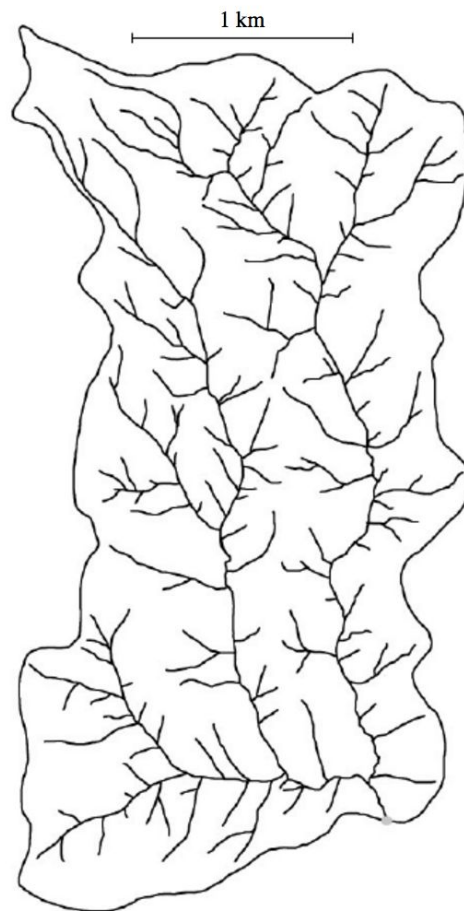


Figure 4. Saint-Michel river basin map, France. The Saint-Michel has a Strahler order 5 river basin with area of  $673 \times 10^4 \text{ m}^2$

Table I. Geomorphological characteristics of the river basins studied

River basin	Area ( $10^4 \text{ m}^2$ )	Strahler order	$\bar{l}_1$ (m)	$\bar{l}_2$ (m)	$\bar{l}_3$ (m)	$\bar{l}_4$ (m)	$\bar{l}_5$ (m)
Saint-Michel	673	5	149	240	1275	426	329
Bunder	29	3	39	62	329	—	—

The model parameters resulting from the experimental river basins are determined manually with the method described above, and appear in Table I.

## RESULTS AND APPLICATIONS

### *The geomorphological transfer function*

The comparison between the model (Equation (7)) and the experimental data is shown in Figure 5. The model parameters are given in Table I for each river basin, and Equation (7) can thus be compared with the experimental Width function of the river basins.

Note that, even though it is far from being perfect, the analytical function of geomorphological transfer provides the experimental Width function with a relatively good trend. It should be noted that no calibration



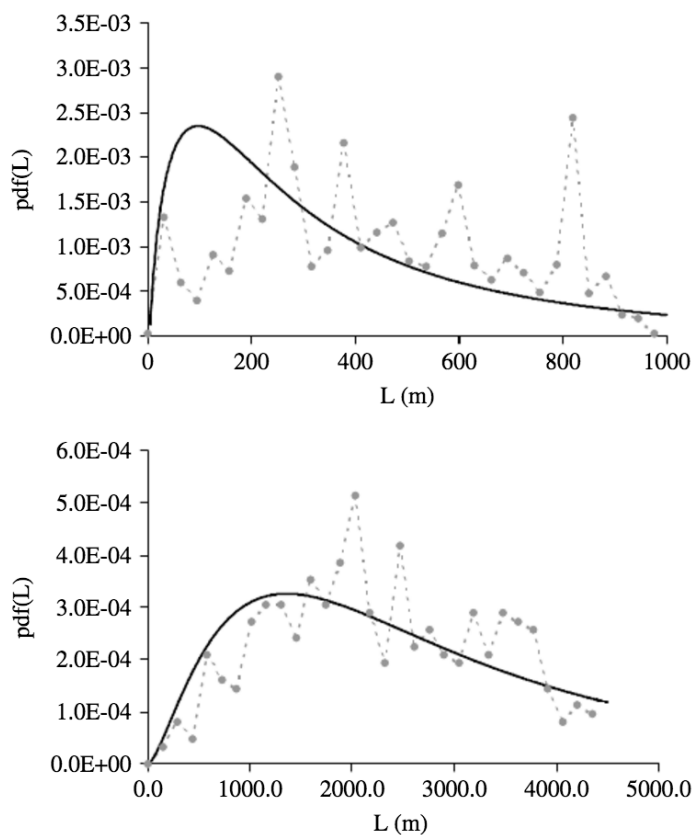


Figure 5. Comparison between the experimental probability density functions of the hydraulic pathways (experimental Width functions) of the drainage basins of Bunder (top, dashed line) and Saint-Michel (bottom, dashed line) and the corresponding theoretical probability density functions (Equation (7)). Simulated parameters are given in Table I

is necessary, since the geomorphological parameters of the transfer function are directly derived from the experimental basins. Here is the major asset of such a function, which is based on general laws on symmetry, and repeats the hydraulic paths distribution (Width function) of the river basin, the geomorphological features of the latter being known (Strahler order and average lengths of hydraulic paths within the different orders).

Moreover, it can be noted that the combination of the symmetry hypotheses with fractal geometry (see model development in appendix), makes it possible to skip scale problems. In fact, although the Saint-Michel river basin has a surface area which is more than twenty times as big as that of Bunder, this scale aspect does not necessarily limit the use of the model.

#### *The AGIUH model*

Equation (2) shows the relation between the GIUH, rainfall and the hydrograph. The function  $T(t)$  corresponds to the distribution of rainfall transfer times over the whole river basin. And yet, the analytical geomorphological transfer function (Equation (7)) specified in the present article is a theoretical Width function; consequently, it determines the lengths distribution covered by rain throughout the whole river basin. It is thus possible to shift from function  $\text{pdf}(L)$  to function  $T(t)$ , supposing (Beven and Wood, 1993) an average runoff speed  $\bar{v}$  in the stream network and  $\bar{v}_h$  in the hillslopes. Then the geomorphological transfer

function (Equation (7)) becomes an AGIUH that reads

$$T(t) = \text{pdf}\left(\frac{L}{\bar{v}}\right) * \text{pdf}\left(\frac{l_0}{\bar{v}_h}\right) \\ = \frac{1}{\sqrt{(2\bar{v})^n \prod_{i=1}^n \bar{l}_i}} \frac{t^{(n/2)-1}}{\Gamma\left(\frac{n}{2}\right)} e^{-\bar{v}t/2\bar{l}_n} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left[\frac{\bar{v}t}{2(\bar{l}_n - \bar{l}_{n-1})}\right]^k * \text{pdf}\left(\frac{l_0}{\bar{v}_h}\right) \quad (9)$$

with  $b_n(k)$  and  $(x)_k$  given in Equation (7).

The aim of this paper is not to discuss the rainfall-runoff modelling. However, it seems important to describe an application of such an AGIUH.

#### Data watershed

To keep as close as possible to the theory of the unit hydrograph, the smaller river basin (Bunder) is chosen in order to fulfil the hypothesis of rainfall homogeneity. The rainfall-runoff data used (Kartiwa, 2004) correspond to two short events. The gross rainfall data are assessed with a tipping-bucket rain-gauge, allowing measurements with an accuracy to within 0.2 mm. Measurements are recorded every 6 min. Water depths at the river outlet are measured with an ultrasound sensor, which gives the distance between the sensor and the depth of runoff. The transformation of water depth into runoff was carried out with a rating curve that had been previously established.

The region is situated in an intertropical zone and has an equatorial type of climate under the influence of a monsoon. It is characterized by an alternation of a rainy season lasting 6 months on average (from November to April) and a dry season starting from May until October. Average annual rainfall is between 1500 and 3000 mm.

#### Production function

In order to determine the net rainfall that transforms gross rainfall into rainfall runoff, it is necessary to assess the runoff deficit. This runoff deficit may be analysed more precisely by studying the infiltration of water into the soil. Several methods are available to assess the infiltration rate of water: Green and Ampt (1911), Horton (1933), Philip (1957) or Holtan (1961). We have chosen Horton's law because, in our present state of knowledge, we believe it is the most appropriate method to explain the hydraulic functioning of our specific case, i.e. the Indonesian drainage basin (Kartiwa, 2004).

This law makes it possible to express the instant capacity for infiltration according to time, in the case of saturating rainfall, in the form of (Horton, 1933)

$$f(t) = f_c + (f_0 - f_c)e^{-kt} \quad (10)$$

where  $f(t)$  ( $\text{mm min}^{-1}$ ) is the infiltration capacity at time  $t$  (min),  $f_0$  ( $\text{mm min}^{-1}$ ) is the initial infiltration capacity,  $f_c$  ( $\text{mm min}^{-1}$ ) is the final infiltration capacity, and  $k$  ( $\text{min}^{-1}$ ) is a constant. The value of the mass infiltration  $F$  is then worked out in the case of saturating rainfall:

$$F(t) = f_c t + \frac{f_0 - f_c}{k} (1 - e^{-kt}) \quad (11)$$

To make the most of Horton's law for modelling, it is necessary to start from the equation linking  $f$  and  $F$  to be able to express (accurately enough) at any time the instant capacity for infiltration according to the amount of water already infiltrated into the ground, whether rainfall is saturating or not:

$$f(t) = f_0 - k(F(t) - f_c t) \quad (12)$$

Net rainfall (RE, rainfall excess) is then worked out using

$$RE(t) = R(t) - [f_0 - k(F(t) - f_c t)] \quad (13)$$

The prevailing types of soils are Mollisols of the haplustolls type on the one hand and haplustolls and Inceptisols of the utropets and ustropets type on the other hand. Land use is mainly gardens (37.8%), agroforestry (30%), and rice fields/groundnut cultivation (24%); 8.2% of the basin's surface are residential areas. The infiltration rate of water has been measured using the submersion method (double-ring infiltrometer; Hills, 1970). The measuring points are chosen for each type of land use. After setting the measured infiltration with the equation of Horton's law, the three parameters  $f_0$ ,  $f_c$  and  $k$  of Horton's law are determined (Table II).

#### Model parameters

The hydraulic pathway averages within the various orders are  $\bar{l}_1 = 39$  m,  $\bar{l}_2 = 68$  m and  $\bar{l}_3 = 336$  m, which represents an average of  $\bar{L} = 443$  m for the whole river network.

The theoretical probability density function of the hillslope hydraulic pathways  $\text{pdf}(l_0) = \Gamma(l_0, \alpha_h, \beta_h)$  is adjusted to experimental data (Figure 2). The shape parameter  $\alpha_h$  is 1.6 and matches the observations of van der Tak and Bras (1990).

The average stream flow velocity in the river network has been determined (Kartiwa, 2004) in the field with a pygmy current meter. The value of the hillslope velocity is calculated using Manning's semi-empirical equation (Chow, 1959):

$$\bar{v}_h = \frac{D^{2/3} S^{1/2}}{n_m} \quad (14)$$

where  $D$  is the average depth of runoff on the hillslope and  $S$  is the average slope. The experimental values of the parameters related to slope appear in Table II. Concerning the average depth of water, the value of parameter  $D$  is very hard to work out. Consequently, we have fixed a value for  $D$  making the value of  $\bar{v}_h$  consistent with the observed runoff rates on the basin slopes studied. The Manning parameter  $n_m = 0.075$  ('heavy brush'). These values lead to  $\bar{v}_h = 0.17$  m s<sup>-1</sup>. The field method provides results in the order of 1 m s<sup>-1</sup> and 0.9 m s<sup>-1</sup> respectively. For the simulations (Figure 6), we have set an average velocity  $\bar{v} = 0.95$  m s<sup>-1</sup> for the stream network.

For each of the two rainfall-runoff events in the Bunder river basin, the net rainfall determined by the production function is convoluted with the AGIUH model following Equation (2), in order to simulate a

Table II. Values of the Horton parameters on the Bunder watershed

Land use	Area (%)	$f_0$ (mm min <sup>-1</sup> )	$f_c$ (mm min <sup>-1</sup> )	$k$ (min <sup>-1</sup> )
Garden	37.8	1.149	0.106	0.244
Agroforestry	30.0	1.335	0.107	0.253
Rice/arachid	24.0	1.278	0.144	0.136
Settlement	8.2	0.490	0.143	0.068
Mean value	—	1.181	0.118	0.206

Table III. Values of the simulation parameters corresponding to the model presented. The model has four common parameters:  $n$ , the Strahler order of the drainage basin;  $\bar{v}$ , the average rate of flow in the stream network;  $\bar{v}_h$ , the average runoff rate on the slopes;  $\bar{l}_0$ , the average hydraulic lengths on the slopes

Model parameter	$n$	$\bar{l}_0$ (m)	$\bar{v}$ (m s <sup>-1</sup> )	$\bar{v}_h$ (m s <sup>-1</sup> )	$\bar{l}_1$ (m)	$\bar{l}_2$ (m)	$\bar{l}_3$ (m)
Value	3	57	0.95	0.17	39	68	336

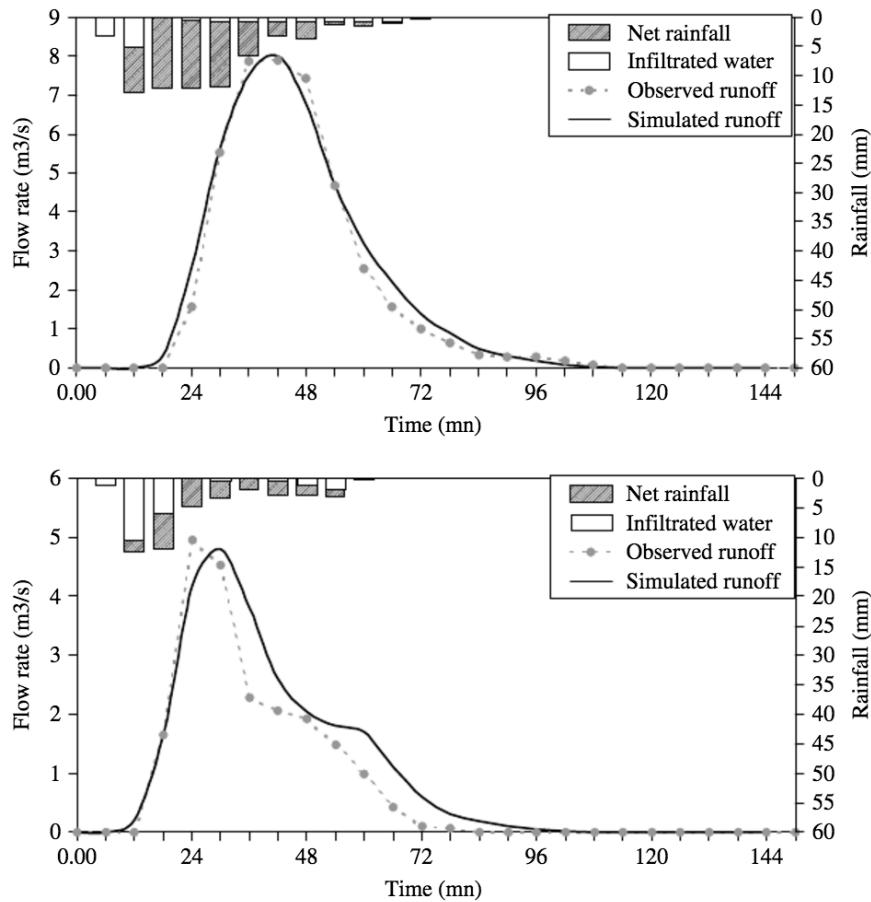


Figure 6. Rainfall-runoff simulation on the Bunder river basin. Observed rainfall-runoff data for the event of 14 November 2001 at Bunder. Infiltration is calculated by an original method from Kartiwa (2004). Simulations are carried out using the AGIUH (Equation (9)) convoluted within net rainfall according to Equation (2). Model parameters are given in Tables I, II and III and the average water velocity on the whole river basin is  $\bar{v} = 0.95 \text{ m s}^{-1}$

hydrograph. This simulated hydrograph is then compared with the real hydrograph (Figure 6). The simulation quality is assessed thanks to the Nash and Sutcliffe (1970) coefficient:

$$F = 1 - \frac{\sum (Q_{\text{exp}} - Q_{\text{sim}})^2}{\sum (Q_{\text{exp}} - \overline{Q_{\text{exp}}})^2} \tag{15}$$

where  $Q_{\text{exp}}$  ( $\text{m}^3 \text{ s}^{-1}$ ) is the actual flow rate,  $Q_{\text{sim}}$  ( $\text{m}^3 \text{ s}^{-1}$ ) is the simulated flow rate and  $\overline{Q_{\text{exp}}}$  ( $\text{m}^3 \text{ s}^{-1}$ ) is the average value of the actual flow rate.

A Nash–Sutcliffe coefficient higher than 0.8 means a good simulation that is close to the stream measurements; this is the case of our simulations, with coefficient values of 0.97 and 0.98.

### CONCLUSION AND PROSPECTS

We have described an analytical geomorphological transfer function representing the theoretical distribution of hydraulic lengths in a river basin. This model is based on the assumption that the reduced lengths distributions  $l_k/R_L$  in order  $k$  (Strahler order) are similar and represented by a Gamma law with  $(1/2, 1/2\bar{l}_k)$  parameters.

The analytical geomorphological transfer function results from the convolution of these  $n$  Gamma laws (Equations (6) and (7)),  $n$  being the total Strahler order in the river basin. This theoretical result is an analytical function depending on  $n + 1$  geomorphological parameters, which are easily available by simply reading a map or analysing digital data (e.g. digital elevation model). These parameters are the Strahler order in the stream network  $n$  and the average lengths  $\bar{l}_1, \bar{l}_2, \dots, \bar{l}_n$  from the drains to the different network orders.

After explaining the approach to obtain this analytical function (Equation (7)), we have tested our theoretical model in two very different river basins, as far as their morphological nature and scale were concerned. The results reveal that this theoretical function, equivalent to a Width function by definition, highlights the real structure of both experimental river basins, without any calibration being implemented. Such function makes it possible to give a fairly accurate description of the morphological structure of the stream network in the river basin.

Then, we presented an applied example of this analytical geomorphological transfer function for the rainfall-runoff modelling. In fact, by assuming that flow velocity is homogeneous through the stream network, it is possible to convert this analytical geomorphological transfer function into an AGIUH (Equation (9)), and thus to model the conversion of rainfall into runoff in a river basin.

The major conclusions that can be drawn are as follows:

- We have described the development of a new AGIUH model, whose geomorphological parameters can be easily and rapidly determined.
- The transfer function, based on the fractal features of stream networks, can be used at multiple scales. This makes it possible to validate the model on river basins having quite different scales.
- The hypotheses on symmetry (Duchesne *et al.*, 1997), which constitute the foundations of the model, make it universal and versatile. In this particular case, the model provides a description of the geomorphometric structure of stream networks, but it may be used for branched structures in general (Duchesne *et al.*, 2002; Fleurant *et al.*, 2004).
- Since the analytical model of the transfer function is parameterized with geomorphological variables derived from the real river basin, the corresponding AGIUH makes it possible to carry out rainfall-runoff simulations through a simple convolution with rainfall and with no calibration. Consequently, the model proves to be a useful tool for anticipating floods and studying the resulting hydrological response even in ungauged river basins.

These findings pave the way for additional validation work of the model, at various scales and with regard to even more varied hydrological applications. Actually, the relation that exists between the river basin's geomorphology and the rainfall-runoff response is a significant issue for the understanding of hydro-geomorphological processes (Poole *et al.*, 2002; Yair and Raz-Yassif, 2004).

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## APPENDIX A

*Detailed calculations concerning the present model*

The determination of both pdf( $l_k$ ) and pdf( $L$ ) needs some mathematical developments, which are presented in this section.

The distribution model of the  $l_k$  drains' hydraulic lengths (Equation (5)) is based on two hypotheses on symmetry, which are combined with the fractal nature of the stream networks (Duchesne *et al.*, 1997; Cudennec, 2000; Cudennec *et al.*, 2004):

- The independence hypothesis of the distribution function of components reduced by the morphometric length  $l_k/R_L$ . According to this hypothesis, the distribution function of a component  $l_k$  only depends on  $l_k$ , not on any of the other components  $l_j$ ,  $k$  being different from  $j$ .
- The isotropy hypothesis of distributions. One considers that the distribution law of  $l_k$  is isotropic. Because of the self-similarity nature of tree structure and Horton's law, the  $k$ -order component is larger, on average, than the  $(k-1)$ -order component, because  $\bar{l}_k = R_L \bar{l}_{k-1}$ . The isotropy hypothesis should then be applied, with the reduced components of branch lengths  $l_k/R_L^{k-1}$  as coordinates of the symbolic space.

Such hypotheses lead to the distribution the  $l_k$  drains' hydraulic lengths (Duchesne *et al.*, 1997; Cudennec, 2000; Cudennec *et al.*, 2004):

$$\text{pdf}(l_k) = \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \frac{1}{\sqrt{l_k}} e^{-\lambda l_k/R_L^{k-1}} \quad (\text{A.1})$$

The constant  $\lambda$  may be explained by expressing the  $l_k$  components average, i.e.  $\bar{l}_k$ , giving successively:

$$\begin{aligned} \bar{l}_k &= \int_0^{+\infty} l_k \text{pdf}(l_k) dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \int_0^{+\infty} \frac{l_k}{\sqrt{l_k}} e^{-\lambda l_k/R_L^{k-1}} dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \int_0^{+\infty} l_k^{1/2} e^{-\lambda l_k/R_L^{k-1}} dl_k \\ &= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\lambda}{R_L^{k-1}}} \Gamma\left(\frac{1}{2} + 1\right) \left(\frac{R_L^{k-1}}{\lambda}\right)^{3/2} \\ &= \frac{R_L^{k-1}}{2\lambda} \end{aligned}$$

After replacing this result in Equation (A.1) we have

$$\text{pdf}(l_k) = \frac{1}{\sqrt{2\pi \bar{l}_k}} \frac{1}{\sqrt{l_k}} e^{-l_k/2\bar{l}_k} \quad (\text{A.2})$$

A Gamma distribution with  $\alpha = 1/2$  and  $\beta = 1/2\bar{l}_k$  can be recognized here, i.e.  $\Gamma(l_k, 1/2, 1/2\bar{l}_k)$ .

The second step consists of determining the probability density distribution for  $L$ , knowing that  $L = \sum_{k=1}^n l_k$ . The usual way to calculate the distribution pdf( $L$ ) is to use the following property (Feller, 1971): if we have  $n$  independent random variables  $l_k$ ,  $k = 1, 2, \dots, n$ , of probability density function pdf( $l_k$ ), then the random variable  $L$  defined as  $L = \sum_{k=1}^n l_k$  has a probability density function pdf( $L$ ) such as

$$\text{pdf}(L) = \text{pdf}(l_1) * \text{pdf}(l_2) * \dots * \text{pdf}(l_n) \quad (\text{A.3})$$

where  $*$  is the convolution integral. To clarify this complex mathematical process, consider the simplest example of what a convolution integral with only two functions is. If  $f(x)$  and  $g(x)$  are two functions:

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(y)g(x - y)dy$$

*Convolution integral of two Gamma distributions*

Let us work out the convolution integral of two Gamma distributions with  $\beta$  parameters that are different in pairs; we then try to calculate  $\Gamma(\alpha, \beta_1) * \Gamma(\alpha, \beta_2)$ . To do this calculation, we will use the characteristic Gamma distribution functions. In fact, to convolute both Gamma distributions means that we are looking for the probability density function pdf( $L$ ) for  $L$  such as  $L = l_1 + l_2$ , where  $l_1$  and  $l_2$  have a probability density function represented by the functions  $\Gamma(\alpha, \beta_1)$  and  $\Gamma(\alpha, \beta_2)$ . Moreover, the characteristic functions  $\varphi_{l_k}^*$  have the following property:

$$\varphi_{l_1+l_2}^* = \varphi_{l_1}^* \varphi_{l_2}^* \tag{A.4}$$

The characteristic function of a random variable  $l_k$  variables here is defined as a mathematical expectation of the exponential function of this random variable. In the present case, therefore, we have

$$\begin{aligned} \varphi_{l_k}^*(s) &= E [e^{-sl_k}] \\ &= \int_0^{+\infty} e^{-sl_k} \text{pdf}(l_k) dl_k \\ &= \frac{1}{\Gamma(\alpha)\beta_k^\alpha} \int_0^{+\infty} l_k^{\alpha-1} e^{-l_k[s+(1/\beta_k)]} dl_k \\ &= \frac{1}{(1 + s\beta_k)^\alpha} \end{aligned}$$

Here one has

$$\varphi_{l_1+l_2}^*(s) = \frac{1}{[(1 + s\beta_1)(1 + s\beta_2)]^\alpha}$$

Concerning our case,  $\alpha = 1/2$ , then function pdf( $L$ ) such as  $L = l_1 + l_2$  can be worked out thanks to Laplace transform tables:

$$\text{pdf}(L) = \frac{1}{\sqrt{\beta_1\beta_2}} \exp \left[ -\frac{\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)L}{2} \right] I_0 \left[ \frac{\left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)L}{2} \right] \tag{A.5}$$

where

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \frac{x^6}{2^2 4^2 6^2} + \dots$$

is the modified Bessel function of the first kind and zero order. Therefore, the result of the probability density function of morphometric lengths  $L$  is

$$\text{pdf}(L) = \frac{1}{2\sqrt{l_1 l_2}} \exp \left[ -\frac{\left(\frac{1}{l_1} + \frac{1}{l_2}\right)L}{4} \right] I_0 \left[ \frac{\left(\frac{1}{l_2} - \frac{1}{l_1}\right)L}{4} \right] \tag{A.6}$$



### Convolution integral of $n$ Gamma distributions

The result of the convolution integral for  $n$  Gamma distributions is obtained by working out the generalized characteristic functions (Mathai, 1982; Moschopoulos, 1985; Sim, 1992). Let  $l_1, l_2, \dots, l_n$ , be  $n$  independent random distributed according to a Gamma distribution of the following type:

$$\Gamma(l, \alpha_i, \beta_i) = \frac{\beta_i^{\alpha_i} l^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\beta_i l} \quad (\text{A.7})$$

Then the probability density function for  $L = \sum_{i=1}^n l_i$  is

$$\text{pdf}(L) = \left( \prod_{i=1}^n \beta_i^{\alpha_i} \right) \frac{L^{\sum_{i=1}^{n-1} \alpha_i - 1}}{\Gamma\left(\sum_{i=1}^n \alpha_i\right)} e^{-\beta_n L} \sum_{k=0}^{+\infty} \frac{b_n(k) \left(\sum_{i=1}^{n-1} \alpha_i\right)_k}{k! \left(\sum_{i=1}^n \alpha_i\right)_k} [(\beta_n - \beta_{n-1})L]^k \quad (\text{A.8})$$

where

$$b_i(k) = \begin{cases} 1 & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\sum_{p=1}^{i-2} \alpha_p\right)_j (-k)_j}{j! \left(\sum_{p=1}^{i-1} \alpha_p\right)_j} C_i^j & \text{for } i = 3, 4, \dots, n \end{cases}$$

with

$$C_i = \frac{\beta_{i-2} - \beta_{i-1}}{\beta_i - \beta_{i-1}} \quad (\text{A.9})$$

and

$$(x)_k = x(x+1)(x+2)\dots(x+k-1) \quad (\text{A.10})$$

By returning to our initial notations, the probability density function of morphometric lengths is defined by

$$\text{pdf}(L) = \frac{1}{\sqrt{2^n \prod_{i=1}^n l_i}} \frac{L^{(n/2)+1}}{\Gamma\left(\frac{n}{2}\right)} e^{-L/2\bar{l}_n} \sum_{k=0}^{\infty} \frac{b_n(k) \left(\frac{n-1}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} \left[ \frac{L}{2(\bar{l}_n - l_{n-1})} \right]^k \quad (\text{A.11})$$

with

$$b_i(k) = \begin{cases} 1 & \text{if } i = 2 \\ \sum_{j=0}^k \frac{b_{i-1}(j) \left(\frac{i-2}{2}\right)_j (-k)_j}{j! \left(\frac{i-1}{2}\right)_j} \left( \frac{\bar{l}_{i-1} - \bar{l}_{i-2} \bar{l}_{i-1}}{\bar{l}_{i-1} - \bar{l}_i \bar{l}_{i-2}} \right)^j & \text{for } i = 3, 4, \dots, n \end{cases}$$