THE FORM OF THE INSTANTANEOUS UNIT HYDROGRAPH

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SUMMARY

An equation is derived for the instantaneous unit hydrograph by assuming that the operation performed by the catchment on the effective rainfall is analogous to that performed by routing through a series of linear reservoirs. A method by which the «best fit» instantaneous unit hydrograph of this form can be derived from a complex flood is demonstrated.

INTRODUCTION

One of the difficulties in obtaining correlations between unit hydrographs and the characteristics of the catchments for which they were derived is the difficulty of expressing the unit hydrograph by the values of one or two parameters. A frequent choice is the peak value of the unit hydrograph of a stated duration (usually the instantaneous unit hydrograph). The obvious disadvantage of this parameter is that the correlation obtained will furnish only the peak of the unit hydrograph of the same period when applied to a catchment whose characteristics are known. A further difficulty is that in preparing the correlation unit hydrographs must be obtained for the chosen period, unless the period itself be taken as an independent variable. Theoretically unit hydrographs of any period can be derived from any given unit hydrograph but the practical difficulty of obtaining a short period unit hydrograph from a long period one is only too well known. This difficulty is due to the fact that small errors in the given unit hydrograph can cause «hunting» which may result in a very distorted shape in the derived short period unit hydrograph. The application of the method of moments or other statistical methods, in itself, will not help. Unless the shape of the short period unit hydrograph is itself restricted in some way any such method will only get closer and closer to the mathematically derived unit hydrograph the more perfect the method is. This problem is also the problem of deriving a short period unit hydrograph from a complex flood. The more perfect the method the more distorted the resulting short period unit hydrograph. In practise a restraint is often imposed on the unit hydrograph shape by smoothing out irregularities as the unit hydrograph is being derived. However, this is an unwieldy and subjective process.

The purpose of this paper is to show that a general equation, for the instantaneous unit hydrograph, containing two parameters, and of sufficient flexibility to permit the close approximation of any empirically derived instantaneous unit hydrograph can be found.

The equation is derived by assuming that the operation performed by the catchment on an instantaneous rainfall is equivalent to a succession of routings through linear storage (1).

The form of the equation is such that the method of moments can be applied to find the «best fit» values of the two parameters even when given only a complex flood.

Notation:

v	=	volume of unit hydrograph	ft^3 hrs/sec
u	=	the oridnate of the unit hydrograph	ft^3/sec
t	=	time	hours
k	=	a parameter having the dimension of time	
n	=	a numerical parameter	
s	=	storage	ft^3 hrs/sec
Q	=	discharge	ft^3/sec
I	=	inflow	ft^3/sec
S.R	=	storm runoff	ft^3/sec
i	=	effective rainfall	ft^3/sec
M_m	=	the m^{th} moment about the origin.	

To derive an equation for the instantaneous unit hydrograph we assume that any catchment may be replaced by a series of n reservoirs each having the storage characteristic $s = k\,Q$ the outflow from one reservoir becoming the inflow to the next. When the instantaneous inflow v takes place to the first reservoir its level is raised by an amount sufficient to accommodate the increased storage and the discharge rises instantaneously from zero to v/k and diminishes with time according to the equation.

$$Q_1 = \frac{v}{k} e^{-t/k} \tag{1}$$

Q_1 becomes the inflow I to the second reservoir and we get (2) an outflow

$$Q_2 = \frac{1}{k} e^{-t/k} \int e^{t/k}\ \mathrm{I}dt$$

$$= \frac{1}{k} e^{-t/k} \int e^{t/k}\ \frac{v}{k}\ e^{-t/k}\ dt$$

$$= \frac{v}{k^2} e^{-t/k}\ t \tag{2}$$

(the constant of integration is zero if we take $Q = 0$ when $t = 0$). Similarly successive routing shows that the outflow from the n^{th} reservoir is given by:

$$Q_n = \frac{v}{k} e^{-t/k} (t/k)^{n-1} / (n-1). \tag{3}$$

This equation is sugested as the general equation of all instantaneous unit hydrographs. To allow of fractional values of n the factorial is replaced by the equivalent gamma function hence:

$$u = \frac{v}{k\,\Gamma(n)} e^{-t/k} (t/k)^{n-1} \tag{3a}$$

This equation in a slightly different form was suggested by Edson [3] on different considerations. The curve has the familiar shape of all short period unit hydrogrpahs. It is shown plotted for different values of n and k in figure 1. k is chosen in each case to produce unit hydrographs of equal peaks.

In order to fit the equation to an empirically derived instantaneous unit hydrograph we shall use the method of moments. As there are two parameters in the equation (n and k) we must calculate the first and second moments of the curve described by equation $3a$.

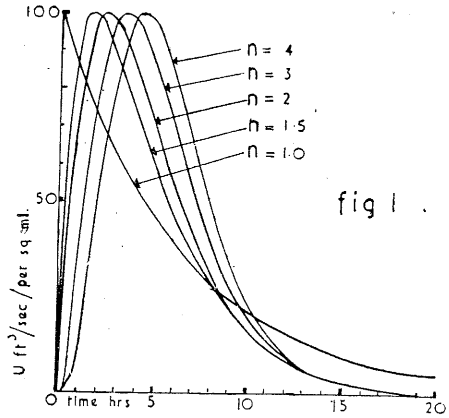


fig 1.

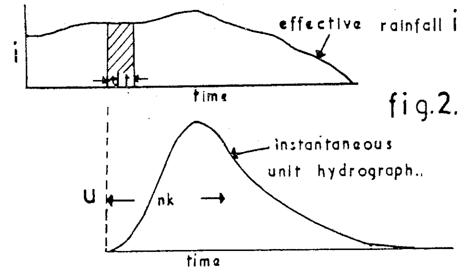


fig.2.

The m^{th} moment about the oigin (Mm)

$$u = \frac{v}{k\,\Gamma(n)}\, e^{-t/k}\, (t/k)^{n-1}$$

$$Mm = \frac{1}{k\,\Gamma(n)} \int_0^\infty e^{-t/k}\, (t/k)^{n-1}\, t^m\, dt$$

$$= \frac{k^m}{\Gamma(n)} \int_0^\infty e^{-t/k}\, (t/k)^{n-1}\, (t/k)^m\, d(t/k)$$

$$= \frac{k^m}{\Gamma(n)} \int_0^\infty e^{-x} x^{n+m-1}\, dx = \frac{k^m}{\Gamma(n)}\, \Gamma(n+m)$$

when $m = 1$ 1st moment $= nk$
$m = 2$ 2nd moment $= n(n+1)\,k^2$ (4)

Equation 4 may be used to obtain the best fit values of n and k for any empirically derived instantaneous unit hydrograph. We shall now discuss how the instantaneous unit hydrograph of best fit, assuming the form of equation 3*a* can be obtained from the records of effective rainfall and storm runoff for any complex flood.

Derivation of Instantaneous Unit Hydrograph

In figure 2 we have an inflow plotted as a fucntion of time. The effect of routing through a series of linear reservoirs is to replace each elementary block *idt* by an elementary outflow $\frac{idt}{k\,\Gamma(n)}\, e^{-t/k}\, (t/k)^{n-1}$ consequently as we have seen the centre of area of each elementary block is moved to the right an amount *nk*, and the centre of area of the total outflow is also moved to the right an amount *nk*

$$\therefore nk = M_1\, S.R - M_1\, i \qquad (5)$$

nk = difference between the first moments of storm runoff and of effective rainfall.

The 2nd moment

Each elementary strip of figure 2 will give rise to an elementary outflow whose 2*nd* moment about its own origin is $n(n+1)k^2$ by equation 4.

Weighting the *2nd* moments of each routed strip with the area of each strip and transferring moments to the origin (i.e. the beginning of effective rainfall) we get:

$$\mathrm{M_2\,S.R} = \frac{1}{\int_0^\infty idt}\left[n(n+1)\;k^2\int_0^\infty idt - n^2k^2\int_0^\infty idt + \int_0^\infty i(t+nk)^2\,dt\right]$$

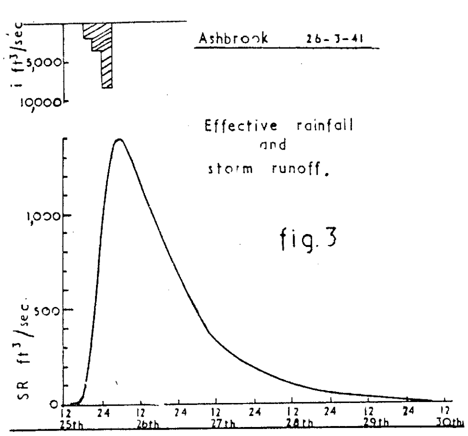
$$= \frac{1}{\int_0^\infty idt}\left[n(n+1)\,k^2\int_0^\infty idt - n^2k^2\int_0^\infty idt + \int_0^\infty (i\,t^2 + 2nkti + n^2k^2 i)\,dt\right]$$

$$= n(n+1)k^2 - n^2k^2 + n^2k^2 + \frac{1}{\int_0^\infty idt}\int_0^\infty i\,t^2\,dt + \frac{2nk}{\int_0^\infty idt}\int_0^\infty i\,t\,dt$$

$$= n(n+1)\,k^2 + \mathrm{M_2}\,i + 2nk\,\mathrm{M}_i\,i$$

$$n(n+1)\,k^2 = \mathrm{M_2\,S.R} - \mathrm{M_2}\,i - 2nk\,\mathrm{M}_i\,i \qquad (6)$$

Equations 5 and 6 enable us to calculate n and k for any catchment from any record of effective rainfall and storm runoff and consequently to derive the instantaneous unit hydrograph of best fit assuming only that it can be described by an equation of the form of $3a$.



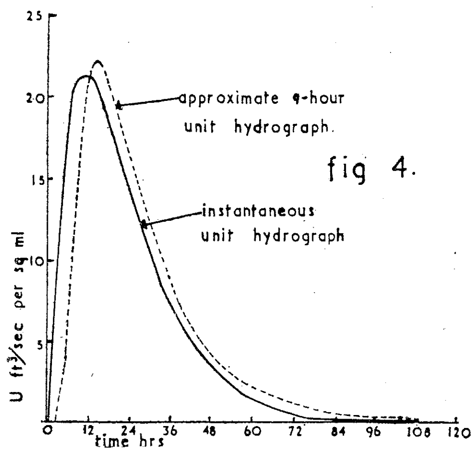


Figure 3 shows a plot of storm runoff and effective rainfall. The latter consists of three 3 hour blocks of different intensities. The subsequent analysis would not be materially changed no matter how complicated the pattern of effective rainfall. The reason for choosing a comparatively short storm is, that as the determination of losses is rather uncertain, we can never be sure of the exact distribution in time of our effective rainfall. In a short storm this is not so important, as the resulting storm runoff will not be seriously affected by the distribution of the effective rainfall within the short period of the total duration of the storm. Consequently in determining the instantaneous unit hydrograph for a complex storm errors due to incorrect determination of the time distribution of rainfall losses will be minimized by confining attention to short storms.

This limitation in no way arises from the method of unit hydrograph analysis presented here but is required solely to ensure the accuracy of our basic data viz: the records of effective rainfall and storm runoff. Table 1 shows the calculation of the required 1*st* and 2*nd* moments. Columns 1, 2 and 3 are self explanatory, column 4 (Δ S.R) gives the volume of storm runoff in each interval. It is expressed in ft^3/sec $\times$ 3 hours and is obtained by taking the mean of S.R ordinates (column 3) at the beginning and end of each interval. Column 5 gives the effective rainfall in each interval also in ft^3/sec $\times$ 3 hours. Column 6 gives twice the time from the beginning of effective rainfall to the centre of each 3 hour interval, expressed in units of 3 hours. Twice the time is used in order to avoid fractional values. Columns 7, 8, 9, 10 give the products required in calculating the 1*st* and 2*nd* moments of the effective rainfall and the storm runoff. The 1*st* and 2*nd* moments of the effective rainfall are calculated by dividing the sums of columns 8 and 10, by the sum of column 5. Similarly the 1*st* and 2*nd* moments of the storm runoff are obtained by dividing the sums of columns 7 and 9 by the sum of column 4. To express the 1*st* and 2*nd* moments in hours and hours squared respectively all first moments are multiplied by 3/2 and all second moments by 9/4.

Applying equations 5 and 6 we get.

$$nk = 28.5 - 5.9 = 22.6$$

$$n(n+1)\,k^2 = 1160 - 40 - 366 = 754$$

from which $n = 2.1$ and $k = 10.8$ hours

Figure 4 shows the instantaneous unit hydrograph described by $n = 2.1$, $k = 10.8$ hrs. For comparison the 9 hour unit hydrograph obtained by assuming the effective rainfall in figure 1 to have been uniformly distributed throughout the 9 hours is also shown. It will be noted that the 9 hour unit hydrograph has a peak greater than that of the instantaneous unit hydrograph. The cause of this discrepancy is that the method of moments tends to give more significance to the extremeties of a distribution than to the centre and consequently the best fit obtained is more in error near the peak than at the extremeties. This is rather infortunate; it can be overcome by using a different fitting method (e.g. least squares) but the labour involved is much greater.

REFERENCES

(1) MASAMI SUGAWARA and FUMIYUKI MARUYAMA. A method of prevision of the river discharge by means of a rainfall model. Symposia Darcy Dijon 1956, vol. 3, p. 71-76.

(2) J. E. NASH and J. P. FARRELL. A graphical solution of the flood routing equation for linear storage-discharge relation. *Trans. A. G. U.* April 1955, Vol. 36, No. 2, p. 319-320.

(3) Charles GRANT EDSON. Parameters for relating Unit Hydrographs to watershed characteristics. *Trans. A. G. U.* August 1951, Vol. 32, No. 4, p. 391-396.

TABLE 1

ASHBROOK			Date of flood. 26-3-41.						
Date	Time	S.R. ft^3/sec	Δ S.R. ft^3/sec × 3*hrs*	*i* ft^3/sec × 3*hrs*	2*x* HRS	2*x* SR	2*x* *i*	$4x^2$ S.R	$4x^2$ *i*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
26th	15	0	15	1829	1	15	1829	15	1829
	18	30	185	3530	3	555	10590	1665	31770
	21	340	660	8330	5	3300	41650	16500	208250
	24	980	1150		7	8050		56350	
	3	1320	1355		9	12195		109755	
	6	1390	1335		11	14685		161535	
	9	1280	1220		13	15860		206180	
27th	12	1160	1100		15	16500		247500	
	15	1040	975		17	16575		281775	
	18	910	850		19	16150		306850	
	21	790	735		21	15435		324135	
	24	680	630		23	14490		333270	
	3	580	530		25	13250		331250	
	6	480	435		27	11745		317115	
	9	390	355		29	10295		298555	
28th	12	320	300		31	9300		288300	
	15	280	260		33	8580		283140	
	18	240	225		35	7875		275625	
	21	210	195		37	7215		266955	
	24	180	167		39	6513		254007	
	3	155	145		41	5945		243745	
	6	135	125		43	5375		231125	
	9	115	108		45	4860		218700	
29th	12	100	92		47	4324		203228	
	15	85	80		49	3920		192080	
	18	70	70		51	3570		182070	
	21	65	62		53	3286		174158	
	24	60	57		55	3135		172425	
	3	55	52		57	2964		168948	
	6	50	48		59	2832		167088	
	9	45	42		61	2562		156282	
30th	12	40	37		63	2331		146853	
	15	35	33		65	2145		139425	
	18	30	28		67	1876		125692	
	21	25	20		69	1380		95220	
	24	15	10		71	710		50410	
	3	5	3		73	219		15987	
31st	6	0							
Totals			13689	13689	//////	260017	54069	7043913	241849
						28.5	5.9	1160	39.8
						1st MSR	1st M*i*	2*nd*MSR	2*nd*M*i*

This paper is a valuable contribution to the problem of finding an objective system for the derivation of unit hydrographs. This discussion is concerned with two points (i) the assumption that a satisfactory instantaneous unit hydrograph can be produced by routing the instantaneous rainfall through a succession of equal linear storages; (ii) the use of moments to find the lag of a complex hydrograph which is also the lag of the instantaneous unit hydrograph.

The author's assumption of a hydrograph produced by successive equal storages is not attractive from a physical viewpoint. The element of translation has been eliminated from the picture and the distribution of the run-off in time based on storage alone. The writer would prefer the assumption of a triangular inflow of base T routed through a linear storage of delay time K. The latter assumption has been discussed by O'Kelly (reference 4), by the author (reference 5) and by the writer (reference 6). It is physically more reasonable, is satisfactory empirically and is mathematically more convenient.

The author's use of moments to determine the lag of the instantaneous unit hydrograph is new to the present writer and is a most valuable suggestion. It can be applied to the unit hydrograph based on triangular inflow since in this case

$$L = \frac{T}{2} + K$$

compared with

$$L = nK$$

in the author's methods. It can be shown that for the unit hydrograph based on triangular inflow the relationship

$$q_p \,.\, L = V$$

is true within the limits of accuracy required. For $T < K$ the error is less than 1% and $T > K$ the error is less than 4%. Thus for any catchment the derivation of the lag as suggested by the author is sufficient to determine the peak of the unit hydrograph no matter what the value of T/K may be. The relation of T to K is only significant in relation to the timing of the peak of the unit hydrograph. To the same degree of accuracy as given above, we can also prove

$$tp = \frac{T}{2}\left(1 + \frac{K}{L}\right)$$

or

$$\frac{tp}{L} = 1 - \frac{K^2}{L^2}$$

The application of these formulae to the author's example results as follows. The lag can be calculated by first moments as in paper to give

$$L = 22.6 \text{ hours.}$$

$$qp = \frac{V}{L}$$

$$= \frac{645}{22.6} \text{ cusecs/sq. mile.}$$

$$= 28.6 \text{ cusecs/sq. mile.}$$

An examination of the recession of the storm as tabulated in Table 1 indicates a K value of 20 hours and hence T value of 5 hours. Figure 5 shows the instantaneous and the 9-hour unit hydrographs for these values of K & T. The agreement with the approximate 9-hour unit hydrograph is better than that obtained by the author's method of successive reservoir routing. The 9-hour hydrograph based on triangular inflow peaking at 25 cusecs/sq. mile and 12 hours.

4. J. J. O'KELLY.
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Institution of Civil Engineers Proceedings, August 1955.
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University College, Galway, 1955.
6. J. C. I. DOOGE.
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Submitted by James C. I. Dooge, Senior Design Engineer, Hydrometric Section, Electricity Supply Board.

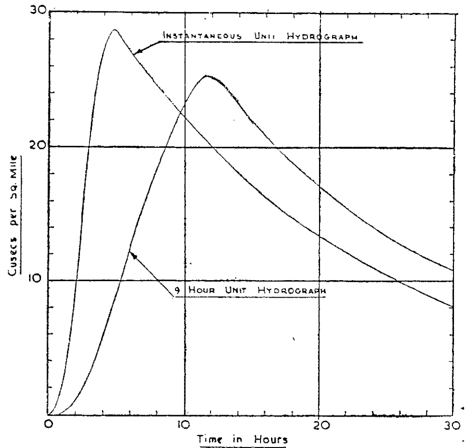

Fig. 5

DISCUSSION

AUTHOR'S CLOSURE

The author is very grateful to Dooge for his suggestions on the use of the method to determine the parameters T and K of the routed triangular inflow. This method is described by Dooge as being «physically more reasonable... satisfactory empirically and mathematically more convenient». The author fully agrees that these three criteria are the important ones but is not convinced that the routed triangle satisfies the criteria better than the successive routing. While there may be some slight advantage in viewing the operation of the catchment as translation and routing, rather than as successive routing, the physical basis of either is rather shaky. It is much too early to say which method is the more satisfactory empirically. The reproduction of a large number of actual unit hydrographs by both methods must be compared before this can be decided. In passing it is perhaps worth noting that the point made by Dooge that the product of the lag and peak of the instantaneous unit hydrograph obtained by routing the triangle is virtually constant indicates a lack of flexibility in the form of the instantaneous unit hydrograph so derived. Clearly a good fit can be got by this method only when the product of the lag and peak of the actual unit hydrograph has this value. The instantaneous unit hydrograph obtained by routing an isosceles triangular inflow of base T through storage S = KQ consists of three parts, viz:— $O < t < T/2$, $T/2 < t < T$, $t > T$, each governed by a different equation. There are, of course, only two parameters, T and K, which can be obtained by equating first and second moments. The equation of the unit hydrograph of period τ contains five such parts. On the other hand, the following equation describes the full unit hydrograph of period τ derived by successive routings.

$$U(\tau, t) = \frac{v}{\Gamma(n)} \left[\Gamma(n, \frac{t}{k}) - \Gamma(n, \frac{t}{k} - \tau)\right]$$

where $\Gamma(n, \frac{t}{k})$ is the ordinate of the incomplete Gamma function of n at $\frac{t}{k}$.

This equation is comparatively simple and being so it enables one to perform many operations mathematically which would otherwise require graphical treatment.