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HYDRODYNAMIC DERIVATION OF STORAGE PARAMETERS OF THE MUSKINGUM MODEL

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ABSTRACT

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The St. Vénant equations for unsteady flow in open channels and the Muskingum method are written both in their conventional forms and in the state-space formulation. The hydrodynamic equation of motion is solved by the method of state trajectory variation and the result for the first-order variation in the state-space variables is used as a basis of linking the parameters of the Muskingum model with the hydraulic parameters of the open channel reach. The results are applicable to any shape of cross-section and to any type of friction law.

TYPES OF DESCRIPTION OF UNSTEADY FLOW IN OPEN CHANNELS

To describe the unsteady motion of water in canals and rivers by means of the methods of mathematical physics, it is necessary to know with sufficient accuracy the geometrical and hydraulic characteristics of the channel reach as well as the initial and boundary conditions. The difficulties of meeting these requirements and the desire to find methods that would be simple, yet sufficiently accurate, led to the development in hydrology of lumped conceptual models and of approaches based on black-box analysis. In the latter cases, the parameters of the model or the ordinates of the system response are based only on input and output data. For channels for which input and output data are not available, these methods can only be used if the parameters can be approximated in some way on the basis of the known geometrical and hydraulic characteristics of the channel reach.

The relationship between the model parameters for the common hydrological methods of flood routing and the hydraulic characteristics of the channel were determined for the special case of a wide rectangular channel with Chézy friction by matching the moments of these conceptual models to the moments of linearised St. Vénant equations (Dooge and Harley, 1967;

Dooge, 1973). In the present paper, the problem of establishing relationships applicable to any shape of cross-section and any friction law is tackled. In particular, a state trajectory variation method is applied to the complete St. Vénant equations (Napiórkowski, 1978) and the result for the first term of the Volterra series is taken as equivalent to the linear form of the Muskingum model in order to determine the relationship between the hydraulic parameters of the St. Vénant equation and the lumped parameters of the Muskingum model.

ST. VÉNANT EQUATIONS FOR UNSTEADY OPEN CHANNEL FLOW

The one-dimensional equation of continuity for unsteady flow in an open channel without lateral inflow is given by:

$$\partial Q/\partial x + \partial A/\partial t = 0 \tag{1a}$$

where $Q(x,t)$ is the discharge and $A(x,t)$ is the cross-sectional area of flow. If the assumption is made that only accelerations in the direction of motion need be taken into account, then the equation for the conservation of linear momentum in this direction can be written (St. Vénant, 1871) as:

$$\frac{\partial z}{\partial x} + \frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{g}\frac{\partial u}{\partial t} + \frac{\tau_0}{\gamma R} = 0 \tag{2a}$$

where $z(x,t)$ is the water surface elevation with respect to a fixed horizontal datum, $u(x,t)$ is the average velocity of flow at the cross-section, τ_0 is the average shear stress along the wetted perimeter of the cross-section, γ is the weight density of water, and R is the hydraulic radius of the cross-section. Nowadays, it is more usual to denote the last term on the left-hand side of eq. 2a as S_f and describe it as the friction slope. In the case where the slope of the channel bed is uniform, it is customary to use the depth of flow rather than the water surface elevation and hence to write the equation of linear momentum as:

$$\frac{\partial y}{\partial x} + \frac{u}{g}\frac{\partial u}{\partial x} + \frac{1}{g}\frac{\partial u}{\partial t} = S_0 - S_f \tag{2b}$$

where $y(x,t)$ is the depth of flow, and S_0 is the bottom slope (a downward slope being taken conventionally as positive).

The problem of flood routing involves the prediction of the hydrograph of flow $Q(t)$, or level $z(t)$, or depth $y(t)$, (or velocity, or area) at the downstream end of a reach on the basis of a given hydrograph of flow or of level at the upstream end. The problem involves the solution of the above set of equations subject to a given initial condition and to appropriate boundary conditions. Since the system of equations is a hyperbolic one, two boundary conditions are required and for the case of tranquil flow (i.e. the Froude number less than 1) one of these boundary conditions must be prescribed at each end of the reach. No analytical solution is available and the problem must be solved either by simplification of the equations or by some method of numerical approximation.

In using numerical methods, the solution is marched forward from one time level to another, using either an explicit or an implicit computation scheme. Thus, before proceeding to make a finite-difference or other approximations the equations are expressed in prognostic form, i.e. the time derivatives are expressed as a function of the space derivatives and of the non-derivative terms. Thus the equation of continuity can be written in prognostic form as:

$$\partial A/\partial t = -\partial Q/\partial x \tag{1b}$$

which is merely a rearrangement of eq. 1a above. Similarly, the dynamic equation can be written in a prognostic form as:

$$\frac{\partial u}{\partial t} = -g\frac{\partial y}{\partial x} - u\frac{\partial u}{\partial x} + g(S_0 - S_f) \tag{2c}$$

by simple rearrangement of eq. 2b.

The second prognostic equation can be written in terms of the same dependent variables as the first (Q and A) by substituting for the velocity $u(x,t)$ in eq. 2c and grouping terms to obtain:

$$\frac{\partial Q}{\partial t} = -g\bar{y}(1-F^2)\frac{\partial A}{\partial x} - \frac{2Q}{A}\frac{\partial Q}{\partial x} + gA(S_0 - S_f) \tag{2d}$$

where $\bar{y}$ is the mean depth of flow defined by:

$$\bar{y} = A/T \tag{3a}$$

and T is the width of the channel at the water surface, i.e.:

$$T = \mathrm{d}A/\mathrm{d}y \tag{3b}$$

F is the Froude number, defined by:

$$F^2 = Q^2 T/gA^3 \tag{4}$$

Eqs. 1b and 2d provide a state transition representation for one-dimensional unsteady flow in an open channel. The state of the system at any time is conveniently represented by the values of the area of flow (A) and the discharge (Q) at all points along the channel at that instant. If the total channel reach is divided into n segments for calculation purposes, then the state space has a dimension of $(2n + 2)$, and input to the system at any instant is given by the terminal conditions of which there will be one at the upstream and one at the downstream end for the case of tranquil flow. From the point of view of the flood routing problem, the output of interest is the value of Q or A downstream in the reach.

The complete dynamic equation given by eq. 2 above represents a distributed model in which the dependent variables are continuous functions of distance along the channel. In many practical problems, data are only available at certain selected points and use is made of black-box analysis or of lumped conceptual models that can be calibrated on the basis of the upstream inflow and the downstream outflow for a reach of river of appreciable length.

In such an approach, the continuous equation of continuity given by eq. 1 is replaced by a lumped continuity equation. This can be derived readily by integrating eq. 1b along the reach:

$$\int_1^2 (\partial A/\partial t)\mathrm{d}x = -\int_1^2 (\partial Q/\partial x)\mathrm{d}x \tag{5a}$$

which can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_1^2 A(x,t)\mathrm{d}x = -[Q(x,t)]_1^2 \tag{5b}$$

which is clearly equivalent to:

$$\mathrm{d}S/\mathrm{d}t = Q_1(t) - Q_2(t) \tag{5c}$$

where $S(t)$ is the storage in the reach, $Q_1(t)$ is the inflow at the upstream end of the reach, and $Q_2(t)$ is the outflow at the downstream end of the reach. Eq. 5c is often referred to as the hydrologic storage equation.

The lumping of the dynamic equation given by eq. 2 is not so readily accomplished. In the classical hydrologic approach to flood routing the dynamic equation is abandoned and replaced by a postulated relationship:

$$S(t) = f[Q_1(t), Q_2(t)] \tag{6}$$

between the three variables in eq. 5c. It should be realised that the operator given on the right-hand side of eq. 6 may be differential in nature as well as algebraic. A relationship of the type given by eq. 6 is sufficient for the solution of the flood routing problem in which we seek to predict the outflow $Q_2(t)$ for a given reach when given the inflow $Q_1(t)$.

In such a lumped formulation of the routing problem, the state-space vector is reduced to a single variable $S(t)$ given by the storage in the reach at any particular instant. Eq. 6, when rearranged to give the outflow $Q_2(t)$ as a function of the state $S(t)$ and the input $Q_1(t)$, gives us the output-state equation in standard form and the insertion of this expression for the output $Q_2(t)$ in eq. 5c gives us the state transition equation in standard form. The differential equation governing the system can be obtained by substituting the expression for the state $S(t)$ given by eq. 6 into eq 5c, thus obtaining on rearrangement:

$$Q_2(t) + \frac{\partial f}{\partial Q_2}\frac{dQ_2}{dt} = Q_1(t) - \frac{\partial f}{\partial Q_1}\frac{dQ_1}{dt} \tag{7}$$

where f is the function defined by eq. 6.

Most of the hydrologic methods of flood routing (at least in their original form) are linear and hence correspond to a linear operator in eq. 6. The simplest linear relationship between storage (S) and outflow (Q_2) is given by:

$$S(t) = KQ_2(t) \tag{8}$$

which represents the basic conceptual model of a linear reservoir with storage delay time K. Such a simple model is found to be inadequate to represent the movement of flood waves in rivers and the next step is to consider replacing this one-parameter relationship by a two-parameter relationship (Sugawara and Maruyama, 1956). If the simple relationship of eq. 8 is generalised by adding a second term involving the time derivative of the outflow (Q_2), then a conceptual model is obtained consisting of two linear reservoirs in series. If, however, the generalisation of two parameters is made by assuming that the storage (S) is linear function of the inflow (Q_1) as well as of the outflow (Q_2), then we obtain the basis relationship of the Muskingum model:

$$S = K[xQ_1(t) + (1-x)Q_2(t)] \tag{9}$$

which was first proposed by McCarthy (1939). When inflow and outflow data are available, they can be used to determine the values of the parameters K and x, and the Muskingum model using these parameters can then be applied to the prediction of outflow due to any given inflow.

The Muskingum method can readily be written in standard state-space form. Rearrangement of eq. 9 gives us the outflow (Q_2) in terms of the state (S) and the input (Q_1) as follows:

$$Q_2(t) = \frac{1}{K(1-x)}S(t) - \frac{x}{1-x}Q_1(t) \tag{10}$$

which is the standard form for the output-state equation for a linear time-invariant system. It will be noted that the input occurs in this output-state equation thus indicating a bypassing of the system, furthermore, that the coefficient of the input is negative thus indicating (for values of x between zero and one) the possibility of negative outputs under certain conditions. The substitution for $Q_2(t)$ in eq. 5c of the expression given on the right-hand side of eq. 10 gives us:

$$\frac{dS}{dt} = -\frac{1}{K(1-x)}S(t) + \frac{1}{1-x}Q_1(t) \tag{11}$$

which is in the standard form of the state transition equation for a linear system.

In the original formulation of the Muskingum method (McCarthy, 1939) and in its early use (Linsley et al., 1948), a physical justification for the basic Muskingum relationship given by eq. 9 was based on the distinction between prism storage and wedge storage in the channel reach. The only reliable way to provide a physical basis for the Muskingum method (and to relate the parameters K and x of the method to the hydraulic parameters in the channel) is to relate eq. 9 to the dynamic equation given in eq. 2. In the remainder of the present paper an attempt is made to relate the Muskingum relationship of eq. 9 to a linearised solution of eq. 2 in order to provide the basis for such a comparison. This linearised solution is obtained by considering only the first-order variations in the state-space variables.

RELATIONSHIP BETWEEN FIRST-ORDER VARIATIONS

The complete non-linear St. Vénant equations can be solved by considering the variations from a steady state-space trajectory (Findeisen et al., 1977; Napiórkowski, 1978). If we consider only the first-order variations then we obtain a linear approximation to the solution of the problem. Such a linear approximation can be compared to the Muskingum model in its linear form in order to obtain an estimate of the parameters of the Muskingum model based on the hydraulic characteristics of the channel.

If we take each variable as consisting of the value for steady uniform flow and the first-order variation from this steady state then we have for the two dependent variables of eqs. 1 and 2d:

$$Q(x,t) = Q_0 + \delta Q(x,t) \qquad \text{and} \qquad A(x,t) = A_0 + \delta A(x,t) \qquad (12\text{a, b})$$

Substituting from eqs. 12a and 12b into the continuity equation (1) gives us the following relationship between the first-order variation in discharge and the first-order variation in area:

$$\frac{\partial}{\partial x}(\delta Q) + \frac{\partial}{\partial t}(\delta A) = 0 \qquad (13)$$

which is identical in form to the original continuity equation given by eq. 1.

The dynamic equation of motion for the original variables can be written in terms of the discharge (Q) and the area flow (A) as:

$$\frac{\partial Q}{\partial t} + g\bar{y}(1-F^2)\frac{\partial A}{\partial x} + \frac{2Q}{A}\frac{\partial Q}{\partial x} = gA(S_0 - S_f) \qquad (14)$$

where $\bar{y}$ is defined by eq. 3a and F by eq. 4. In this discussion the friction slope S_f is taken in the completely general form and may be written as:

$$S_f = f[A, Q, (\text{shape}), (\text{roughness})] \qquad (15)$$

For the reference conditions of steady uniform flow, every term on the l.h.s. of eq. 14 is identically equal to zero because of the assumption of steadiness and uniformity. The r.h.s. of the equation is equal to zero because for steady uniform flow we have:

$$S_f = S_0 \tag{16}$$

In a linear approximation to eq. 14 the dependent variables Q and A will be replaced by their approximations given in eqs. 12a and 12b. The coefficients of the derivative terms of the l.h.s. of eq. 14 will be replaced by their reference values plus the linear variation from it and the latter linear variation will be functions of the linear variations in discharge and in area defined in eq. 12. However, because the reference condition is assumed to be uniform, the terms are all zero under the reference conditions and consequently any variations in the coefficients will not enter into the first-order approximation. On the r.h.s., however, where the terms are not separately equal to zero for the reference condition, the first-order variations will have to be taken into account. Since the friction slope S_f is a function of discharge (Q) and of area (A) then a first-order approximation to the friction slope may be obtained by writing a Taylor's series expansion as follows:

$$S_f = S_0 + (\partial S_f/\partial A)\delta A + (\partial S_f/\partial Q)\delta Q \tag{17}$$

where the derivatives of the friction slope with respect to the discharge and the area are evaluated at the reference condition. When the appropriate substitutions are made from eqs. 12 and 17 we obtain a second relationship between the first-order variations in discharge and in area as follows:

$$\frac{\partial}{\partial t}(\delta Q) + g\bar{y}_0(1 - F_0^2)\frac{\partial}{\partial x}(\delta A) + \frac{2Q_0}{A_0}\frac{\partial}{\partial x}(\delta Q)$$
$$= gA_0\left(-\frac{\partial S_f}{\partial Q}\delta Q - \frac{\partial S_f}{\partial A}\delta A\right) \tag{18}$$

which is the required linearised form of eq. 14.

The task remaining is to use the linearised equation of motion given by eq. 17 as the basis for the scale parameter K and the shape parameter x in the basic linear relationship of the Muskingum model defined by eq. 9. Eq. 17 differs in form in two respects from eq. 9. Firstly, eq. 17 contains derivatives with respect to both space and time (i.e. is prognostic in character) whereas eq. 19 refers only to the relationship between variables at a given time level and is therefore diagnostic in character. The second difference is that eq. 17 represents a distributed model and its solution would be a function of distance along the channel. On the other hand, eq. 9 represents a lumped model in so far as the total storage in the channel reach is expressed as a function only of the flow at each end of the reach. Accordingly, in order to make a direct comparison it is necessary to transform eq. 17 from diagnostic to prognostic form and to transform it from a distributed to a lumped formulation.

In order to reduce eq. 18 to diagnostic form, it is necessary either to neglect completely the term involving the derivative with respect to time of the variation discharge or else to express it in terms of derivatives with respect to distance. The latter course would be preferable if a reasonable basis for such expression can be derived. One convenient method of doing this is to use the kinetic-wave approximation as the basis for the elimination of the time derivative. In this approximation the assumption is made (which is reasonable under most practical conditions) that the two terms on the r.h.s. of eq. 14 are substantially larger than the terms on the l.h.s. and accordingly that the latter terms may be neglected as a first approximation. If this approach is applied to the linearised version in eq. 18 then we set the l.h.s. of that equation equal to zero and obtain as a first approximation:

$$\delta Q = -\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\,\delta A \tag{19}$$

The above expression can now be used to approximate the first term on the l.h.s. of eq. 18 as:

$$\frac{\partial}{\partial t}(\delta Q) = -\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\frac{\partial}{\partial t}(\delta A) \tag{20a}$$

which by combination with the equation of continuity given by eq. 13 can be written as:

$$\frac{\partial}{\partial t}(\delta Q) = \frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\frac{\partial}{\partial x}(\delta Q) \tag{20b}$$

Substituting from eq. 20b into eq. 18 and gathering terms we obtain:

$$g\bar{y}_0(1-F_0^2)\frac{\partial}{\partial x}(\delta A) + \left(\frac{2Q_0}{A_0} + \frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\right)\frac{\partial}{\partial x}(\delta Q) = gA_0\left(-\frac{\partial S_f}{\partial Q}\,\delta Q - \frac{\partial S_f}{\partial A}\,\delta A\right) \tag{21}$$

which is completely diagnostic in form, i.e. it relates only to the relationship between the variables at a given time level.

The relationship between the first-order variation in discharge and in area given by eq. 21 can be reduced to a lumped form capable of comparison with the Muskingum relationship of eq. 9 by writing the equation in terms of the values of these variables at the two ends of the reach. If our only knowledge of the area of flow is at the upstream and downstream ends then we are forced to approximate the space derivative of the variation in area which occurs in the first term on the l.h.s. of eq. 21 as:

$$\frac{\partial}{\partial x}(\delta A) = \frac{\delta A_2 - \delta A_1}{L} \tag{22}$$

where A_1 is the area flow at the upstream end of the reach, A_2 is the area at the downstream end of the reach, and L is the length of the reach. Similarly the space derivative of the first-order variation in flow must be written as:

$$\frac{\partial}{\partial x}(\delta Q) = \frac{\delta Q_2 - \delta Q_1}{L} \tag{23}$$

It is clear that the assumptions made in eqs. 22 and 23 are only reasonable in cases of unsteady flow where the wavelength of the flood wave is large compared to the length of the channel reach. For the accurate simulation of flows of shorter wavelengths, a model based on multiple reach lengths would have to be used.

We are now in a position to evaluate eq. 21 at each end of the reach. When writing the equation for the upstream end of the reach at which the area of flow is A_1 and the discharge is Q_1, the terms on the l.h.s. of eq. 21 are evaluated by means of eqs. 22 and 23 while the terms on the r.h.s. of eq. 21 are evaluated at the upstream section so that we can write eq. 21 as:

$$g\bar{y}_0(1-F_0^2)\left(\frac{\delta A_2 - \delta A_1}{L}\right) + \left(\frac{2Q_0}{A_0} + \frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\right)\left(\frac{\delta Q_2 - \delta Q_1}{L}\right) = gA_0\left(-\frac{\partial S_f}{\partial Q}\,\delta Q_1 - \frac{\partial S_f}{\partial A}\,\delta A_1\right) \tag{24a}$$

By collecting the terms involving the areas on one side of the equation and the terms involving the discharges on the other side of the equation we can write:

$$\left[\frac{g\bar{y}_0}{L}(1-F_0^2) - gA_0\frac{\partial S_f}{\partial A}\right]\delta A_1 - \left[\frac{g\bar{y}_0}{L}(1-F_0^2)\right]\delta A_2 = \left[gA_0\frac{\partial S_f}{\partial Q} - \frac{2Q_0}{A_0 L} - \frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\frac{1}{L}\right]\delta Q_1 + \left[\frac{2Q_0}{A_0 L} + \frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\frac{1}{L}\right]\delta Q_2 \tag{24b}$$

which for convenience can be written as:

$$(M-N)\delta A_1 - M\delta A_2 = (P-R)\delta Q_1 + R\delta Q_2 \tag{24c}$$

where the values of M, N, P and R are respectively given by:

$$M = \frac{g\bar{y}_0}{L}(1-F_0^2), \qquad N = gA_0(\partial S_f/\partial A) \tag{25a, b}$$

$$P = gA_0(\partial S_f/\partial Q) \qquad \text{and} \qquad R = \frac{2Q_0}{A_0 L} + \frac{1}{L}\frac{\partial S_f/\partial A}{\partial S_f/\partial Q} \tag{25c, d}$$

Eq. 24 gives us a relationship between the first-order variations in discharge and area at the two ends of the reach based on the application of eq. 21 to conditions at the upstream end of the reach.

A similar equation can be derived for the downstream end of the reach. When eq. 21 is written for the downstream end of the reach, the derivative terms on the l.h.s. of the equation are again approximated by eqs. 22 and 23 and the variations on the r.h.s. of eq. 21 are written for conditions at the downstream section where the variation in discharge is Q_2 and the variation in area is A_2. Accordingly, we have for the downstream end of the reach:

$$g\bar{y}_0(1-F_0^2)\left(\frac{\delta A_2-\delta A_1}{L}\right)+\left(\frac{2Q_0}{A_0}+\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\right)\left(\frac{\delta Q_2-\delta Q_1}{L}\right)$$
$$=gA_0\left(-\frac{\partial S_f}{\partial Q}\,\delta Q_2-\frac{\partial S_f}{\partial A}\,\delta A_2\right) \tag{26a}$$

in which the terms can again be gathered to give:

$$\left[\frac{g\bar{y}_0(1-F_0^2)}{L}\right]\delta A_1-\left[\frac{g\bar{y}_0}{L}(1-F_0^2)+gA_0\,\frac{\partial S_f}{\partial A}\right]\delta A_2$$
$$=-\left[\frac{2Q_0}{A_0L}+\frac{1}{L}\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\right]\delta Q_1+\left[gA_0\,\frac{\partial S_f}{\partial Q}+\frac{2Q_0}{A_0L}+\frac{1}{L}\,\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\right]\delta Q_2 \tag{26b}$$

which in turn can be written as:

$$M\delta A_1-(M+N)\delta A_2=-R\delta Q_1+(P+R)\delta Q_2 \tag{26c}$$

in which the parameters M, N, P and R have the same meaning as in the respective eqs. 25a—25d and have the same values as in eq. 24c since they are all evaluated at the reference conditions.

We have now obtained two equations involving the first-order variation in the flow and in the area at the two ends of a channel reach. For the upstream end we have:

$$(M-N)\delta A_1-M\delta A_2=(P-R)\delta Q_1+R\delta Q_2 \tag{24c}$$

while for the downstream end of the reach we have:

$$M\delta A_1-(M+N)\delta A_2=-R\delta Q_1+(P+R)\delta Q_2 \tag{26c}$$

These two linear equations can readily be solved in order to express the variation in area at either end of the reach in terms of the discharge at each end of the reach. Thus eliminating the variation in area at the downstream end of the reach δA_2 we obtain:

$$N^2\delta A_1=-[(M)(R)+(M+N)(P-R)]\delta Q_1$$
$$+[(M)(P+R)-(M+N)(R)]\delta Q_2 \tag{27}$$

By substituting for M, N, P and R from the respective eqs. 25a—25d we obtain an explicit expression for the first-order variation in the area of flow at the upstream end of the channel in terms of the first-order variation of flow at the upstream end of the channel (δQ_1), the first-order variation of flow at the downstream end of the channel (δQ_2) and the hydraulic parameters of the channel at the reference discharge. Similarly, we can obtain an explicit expression for the first-order variation in the area of flow at the downstream end of the channel by eliminating the linear variation of flow at the upstream end of the channel (δA_1) from eqs. 24c and 26c above. When this is done we obtain:

$$N^2 \delta A_2 = - [(M-N)(R) + (M)(P-R)]\delta Q_1 + [(M-N)(P+R) - (M)(R)]\delta Q_2 \quad (28)$$

Eqs. 27 and 28 are expressions for the first-order variation of the area of flow at the upstream and downstream end of the reach, respectively, as linear functions of the variation in discharge at the upstream and downstream end of the reach. These relationships can be used as the basis for comparison with the basic Muskingum relationship of eq. 9.

ESTIMATION OF MUSKINGUM PARAMETERS

The basic assumption of the Muskingum method is usually written as:

$$S(t) = K[xQ_1(t) + (1-x)Q_2(t)] \quad (9)$$

where K and x are parameters to be determined in some way. If the parameters are determined on the basis of eqs. 27 and 28, derived in the last section, then these values will apply in the neighbourhood of the reference condition about which the variation in eqs. 27 and 28 are taken. For the use of the Muskingum method as a linear model then the same values of the parameters would be used throughout the whole range of flow. In the case of the use of the Muskingum method as a non-linear model in which the parameters K and x would vary with discharge then the values can still be estimated on the basis of eqs. 27 and 28 for any number of reference discharges and the variation of the parameters with reference discharge determined.

For the Muskingum model the variation of storage in the reach is given by:

$$\delta S = K[x\delta Q_1 + (1-x)\delta Q_2] \quad (29)$$

If the length of the reach is small compared to the wavelength of the unsteady flow (as was assumed in the last section in eqs. 22 and 23), then the storage in the reach can be approximated by:

$$\delta S = \tfrac{1}{2}L(\delta A_1 + \delta A_2) \quad (30)$$

at any instant of time. The relationship between the first-order variation in storage and the first-order variations in area at each end of the reach given by eq. 30 can be transformed to a relationship between the variation in storage and the variation in discharge at the ends of the reach by substituting from eqs. 28 and 28 into eq. 30. When this is done we obtain:

$$\frac{2N^2}{L}\,\delta S = [-(2M+N)P+2NR]\,\delta Q_1 + [(2M-N)P-2NR]\,\delta Q_2 \qquad (31)$$

which is the rleationship between the first-order variation in storage in the reach and the first-order variation in discharge at the two ends of the reach, based entirely on the linearisation and the lumping of the hydrodynamic equation of motion.

Comparison of the coefficients of corresponding terms in eqs. 29 and 31 allows us to write:

$$Kx = \frac{L}{2N^2}[-(2M+N)P+2NR] \qquad (32)$$

for the coefficient of the first-order variation of the discharge at the upstream end of the reach and to write:

$$K(1-x) = \frac{L}{2N^2}[(2M-N)P-2NR] \qquad (33)$$

for the coefficient of the variation in discharge at the downstream end of the reach. By adding eqs. 32 and 33 we get the hydrodynamic estimate of the scale parameter K as:

$$K = \frac{L}{2N^2}(-2NP) \qquad (34a)$$

which reduces to:

$$K = -\frac{P}{N}L \qquad (34b)$$

Substitution from eqs. 25b and 25c gives us the scale parameter K of the Muskingum model in terms of the hydraulic parameters as:

$$K = \left(-\frac{\partial S_f}{\partial Q}\Big/\frac{\partial S_f}{\partial A}\right)L \qquad (34c)$$

so that this parameter depends only on the length of channel and on the particular form of the general friction law appropriate to the flow being considered. Since the derivatives of the friction slope (S_f) with respect to discharge (Q) and area of flow (A) are evaluated for the steady uniform reference conditions the total derivative in the friction slope must be zero so that we can write:

$$-\frac{\partial S_f}{\partial A}\Big/\frac{\partial S_f}{\partial Q} = \frac{dQ}{dA} = c_k \qquad (35)$$

where c_k is the kinematic wave speed (Lighthill and Whitham, 1955). Combining eqs. 34c and 35 we have:

$$K = L/c_k \qquad (36)$$

Accordingly, the scale parameter (K) of the Muskingum model is seen as the time taken to traverse the channel at the kinematic wave speed. Since long waves (for which the model is adequate) travel at approximately the kinematic wave speed, the Muskingum scale parameter (K) is identified with the time of travel of long waves through the channel reach under consideration.

The second parameter x of the Muskingum relationship can be determined from eqs. 32 and 34. Dividing the former equation by the latter we obtain:

$$x = \frac{1}{2NP}\left[(2M+N)P - 2NR\right] \tag{37a}$$

which can be written as:

$$x = \tfrac{1}{2} + M/N - R/P \tag{37b}$$

Substitution from eqs. 25a and 25b gives us:

$$\frac{M}{N} = \frac{1}{\partial S_f/\partial A}\,\frac{\bar{y}_0}{A_0 L}\,(1 - F_0^2) \tag{38}$$

Similarly, substitution from eqs. 25c and 25d gives us:

$$\frac{R}{P} = \frac{1}{\partial S_f/\partial Q}\,\frac{2Q_0}{gA_0^2 L} + \frac{\partial S_f/\partial A}{(\partial S_f/\partial Q)^2}\,\frac{1}{gA_0 L} \tag{39a}$$

In order to simplify the expression for the shape parameter (x) it is convenient to write:

$$m = -\frac{\partial S_f/\partial A}{\partial S_f/\partial Q}\,\frac{A_0}{Q_0} \tag{40}$$

in which the parameter m is the ratio of the kinematic wave speed given by eq. 35 to the average velocity of flow at the reference conditions. Using eq. 40, we can write eq. 39a as:

$$\frac{R}{P} = -\frac{1}{\partial S_f/\partial A}\,m\,\frac{2Q_0^2}{gA_0^3 L} + \frac{1}{\partial S_f/\partial A}\,m^2\,\frac{Q_0^2}{gA_0^3 L} \tag{39b}$$

or

$$\frac{R}{P} = -\frac{1}{\partial S_f/\partial A}\,\frac{Q_0^2}{gA_0^3 L}\,(2m - m^2) \tag{39c}$$

using the definition of the Froude number (F) in eq. 4 this can be written as:

$$\frac{R}{P} = -\frac{1}{\partial S_f/\partial A}\,\frac{\bar{y}_0}{A_0 L}\,F_0^2(2m - m^2) \tag{39d}$$

Substituting from eq. 38 and 39d in eq. 37b we obtain the expression:

$$x = \tfrac{1}{2} + \frac{1}{\partial S_f/\partial A}\,\frac{\bar{y}_0}{A_0 L}\left[1 - (m-1)^2 F_0^2\right] \tag{37c}$$

which gives us the estimation of the Muskingum shape parameter x on the basis of the hydraulic parameters of the channel at the reference conditions. It should be noted that the friction slope varies inversely with the area of flow so that the value of the parameter x will be less than 0.5 even for a Froude number of 1.0 as long as the value of the parameter m defined by eq. 40 is less than 2.0.

APPLICATION TO SPECIAL FRICTION LAWS AND CHANNEL SHAPES

The significance of eqs. 36 and 40 can readily be seen if we apply the results to the friction laws in general use and to some particular shapes of channel. Bakhmeteff (1932) suggested that for most of the shapes commonly encountered in open channel flow, the friction slope for rough turbulent flow could be taken as:

$$S_f = (\text{constant}) \times (Q^2/y^{2n}) \tag{41}$$

For this particular relationship we would have:

$$\partial S_f/\partial Q = 2S_0/Q_0 \tag{42}$$

for the variation of the friction slope with discharge at the reference conditions; and:

$$\frac{\partial S_f}{\partial A} = -\frac{2n}{T_0}\frac{S_0}{y_0} \tag{43}$$

where T_0 is the surface width as defined by eq. 3d for the reference conditions, as the expression for the variation of friction slope with area at the same reference condition. If the expressions from eqs. 42 and 43 are substituted in eq. 35 we obtain for the kinematic wave speed:

$$c_k = n\frac{Q_0}{T_0 y_0} = n\frac{\bar{y}_0}{y_0}\frac{Q_0}{A_0} \tag{44}$$

where $\bar{y}_0$ is the mean depth of flow defined by eq. 3a, and y_0 is the full depth of flow used in eq. 41. For any given shape of section, the value of these parameters can readily be determined and the value of the kinematic wave speed given by eq. 44 used in eq. 36 to determine the scale parameter (K) of the Muskingum model. For the same case the value of the parameter m defined by eq. 40 will be given by:

$$m = n(\bar{y}_0/y) \tag{45}$$

and substitution from eqs. 43 and 45 in eq. 40 gives us:

$$x = \frac{1}{2} - \frac{1}{2m}\left(\frac{y_0}{S_0 L}\right)\left[1 - \left(\frac{m\bar{y}_0}{y_0} - 1\right)^2 F_0^2\right] \tag{46}$$

which is the general expression for the shape parameter x for friction relationship of the type given by eq. 41.

For the case of a rectangular channel the mean depth ($\bar{y}_0$) is equal to the actual depth (y_0). Accordingly, we will have for the scale parameter:

$$K = L/mu_0 \tag{47}$$

where u_0 is the average velocity at the reference conditions and n is the parameter in eq. 41 which will depend on the friction law. For the case of Chézy friction we will have:

$$m = \tfrac{3}{2} \tag{48}$$

so that for a rectangular channel with Chézy friction the scale parameter in the Muskingum model will be given by:

$$K = \tfrac{2}{3}(L/u_0) \tag{49}$$

For this case (rectangular channel with Chézy friction) the shape parameter of eq. 46 can be given by:

$$x = \tfrac{1}{2} - \tfrac{1}{3}(y_0/S_0L)(1 - \tfrac{1}{4}F_0^2) \tag{50}$$

The values of K and x in eq. 49 and 50 are identical to those obtained for this special case of a rectangular channel with Chézy friction by comparing the first and second moments of the impulse response of the Muskingum model with the first and second moments of the linearised channel response, i.e. the impulse response of the linearised version of the complete St. Vénant equation (Dooge, 1973).

The effect of using a different friction law can readily be illustrated for the case of Manning friction in a rectangular channel. If the Manning equation is used then we have:

$$m = \tfrac{5}{3} \tag{51}$$

Thus for a rectangular channel with Manning friction, eq. 47 becomes:

$$K = \tfrac{3}{5}(L/u_0) \tag{52}$$

which may be compared to eq. 49 for the case of a rectangular channel with Chézy friction. The parameter x will also be varied. Insertion with the appropriate value from eq. 51 into eq. 46 gives us for the case of rectangular channel with Manning friction:

$$x = \tfrac{1}{2} - \tfrac{3}{10}(y_0/S_0L)(1 - \tfrac{4}{9}F_0^2) \tag{53}$$

which can be compared with eq. 50 for the case of a rectangular channel with Chézy friction.

For shapes of channel other than wide rectangular, the parameters will take on different values. This may be illustrated for the case of a 90° triangular flume which is quite different in shape from the wide rectangular channel. For such a triangular flume with Chézy friction, the value of the parameter n in eq. 41 is given by:

$$n = \tfrac{5}{2} \tag{54}$$

and of parameter m in eq. 45 is:

$$m = \tfrac{1}{2} n = \tfrac{5}{4} \tag{55}$$

Eq. 36 for the scaling parameter K takes the form:

$$K = \tfrac{4}{5}(L/U_0) \tag{56}$$

and the shape parameter x is given by:

$$x = \tfrac{1}{2} - \tfrac{1}{5}(1 - \tfrac{1}{16} F_0^2) \tag{57}$$

For the same 90° flume with Manning friction we have:

$$n = \tfrac{8}{3} \qquad \text{and} \qquad m = \tfrac{1}{2}\ n = \tfrac{4}{3} \qquad (58),(59)$$

so that the scale parameter is given by:

$$K = \tfrac{3}{4}(L/u_0) \tag{60}$$

and the shape parameter x by:

$$x = \tfrac{1}{2} - \tfrac{3}{16}(1 - \tfrac{4}{9} F_0^2)(y_0/S_0 L) \tag{61}$$

Most of the shapes and friction laws encountered in practice (except for laminar flow) would be expected to be intermediate between a triangle section with Chézy friction whose parameters are given by eqs. 56 and 57 and a wide rectangular channel with Manning friction whose parameters are given by eqs. 52 and 53.

It is clear from eq. 46 that as the dimensionless length of the channel $(S_0 L)/y_0$ becomes greater and greater the value of the shape parameter will approach the value $x = 0.5$ asymptotically. It will never exceed that figure for turbulent flow at Froude numbers less than one except in the exceptional case where the discharge at uniform flow increases less rapidly than the area of flow. While the latter conditions would hold for free surface flow in a circular pipe which was almost full, which is basically unstable, it would not occur in the shapes normally encountered in open channel flow computations. For laminar flow we would have $m = 3.0$ and thus the expression within brackets in eq. 40 could be negative for Froude numbers approaching unity. This would result in a value of x greater than 0.5, which would indicate amplification at all frequencies.

For very small lengths of channel the value of the parameter x as given by eq. 40 could be negative. While such a negative value is difficult to reconcile with the concept of prism storage originally spoken of in connection with the Muskingum method, it is the appropriate value of the parameter for the best fit to the linearised equation.

REFERENCES

Bakhmeteff, B.A., 1932. Hydraulics of Open Channels. McGraw-Hill, New York, N.Y.

Dooge, J.C.I., 1973. Linear theory of hydrologic systems. U.S. Agric. Res. Serv., Tech. Bull. No. 1468.

Dooge, J.C.I. and Harley, B.M., 1967. Linear routing in uniform open channels. Proc. Int. Hydrol. Symp., Fort Collins, Colo. Sept. 1967, Pap. No. 8, 1: 57—63.

Findeisen, W., Szymanowski, J. and Wierzbicki, A., 1977. Theory and computational methods of optimization W.N.T. Warsaw, (in Polish).

Lighthill, M.H. and Whitham, G.B., 1955. On kinematic waves, 1. Flood movements in long rivers. Proc. R. Soc. London, Ser. A, 229: 281—316.

Linsley, R.K., Kohler, M.A. and Paulhus, J.L.H., 1949. Applied Hydrology. McGraw-Hill, New York, N.Y.

McCarthy, G.T., 1939. The unit hydrograph and flood routing. U.S. Corps Eng., Providence, R.I.

Napiórkowski, J.J., 1978. Identification of the conceptual reservoir model described by Volterra series. D.Sc. Thesis, Institute of Geophysics, Polish Academy of Sciences, Warsaw (in Polish).

Saint-Vénant (Barre de), 1971. Theory of unsteady water flow with application to river floods and to propagation of tides in river channels. C.R. Acad. Sci. Paris, 73: 148—154; 237—240 (in French).

Strupczewski, W.G. and Kundzewicz, Z., 1980. Muskingum method revisited. J. Hydrol., 48: 327—342.

Sugawara, M. and Maruyama, F., 1957. A method of provision of the river discharge by means of a rainfall model. Proc. Darcy Symp. Dijon, Int. Assoc. Hydrol. Sci., Publ. No. 42, 3: 71—76.